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A STUDY OF TIME-OPTIMAL INTERCEPT IN TWO DIMENSIONS (Vol. II)

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FOREWORD

This report, consisting of two volumes, was prepared by Aeronautical Research Associates of Princeton, Inc. (ARAP) under USAF Contract Number AF33(657)-11319. The contract was initiated under Project No. 8219, "Stability and Control Investigation", Task No. 821904, "Systems Analysis and Optimization". The work was administered under the direction of Flight Dynamics Laboratory, FDCC, RTD, Wright-Patterson Air Force Base, Ohio, with Mr. Lawrence Schwartz, project engineer until June 1963, and Mr. Ronald Anderson, project engineer from June 1963.

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ABSTRACT

A planar time-optimal formulation is used to study the terminal phase of interception above the atmosphere in a uniform gravitational field. The dependence on initial conditions of the optimal steering angle is examined for initial relative velocities ranging from 5,000 ft/sec to 50,000 ft/sec and initial distances up to 500,000 ft. Results are presented in graphical form for two typical rockets showing: (1) the range dependence of terminal error sensitivities to errors in measurements of initial conditions taken along the initial rectilinear coasting path and (2) the variation in these terminal error coefficients along optimal interception paths. Some interceptions with proportional control systems were made and compared with the optimal paths.

This report has been reviewed and is approved.

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TABLE OF CONTENTS

ILLUSTRATIONS

ILLUSTRATIONS (Cont'd.)

TABLES

LIST OF SYMBOLS

x, y - relative distance components in target centered coordinate frame (ft.)

V_x - relative velocity in x-direction (ft/sec.)

V_y - relative velocity in y-direction (ft/sec.)

V - relative velocity (ft/sec.)

R - relative range (ft/sec.)

ϕ - angle between V and positive x-axis. Measured in positive sense counterclockwise from x to V (deg.)

α - thrust direction angle measured positive counterclockwise from x-axis to thrust vector (deg.)

T - time at which minimum controlled miss distance with perturbed thrust direction angle occurs (sec.)

m_o - initial vehicle mass (total)

β - propellant mass flow rate (slugs/sec.)

c_o - rocket effective exhaust velocity (ft/sec.)

t_b - length of thrusting period (sec.)

ρ_o - minimum uncontrolled miss distance (ft.)

t_o - time at which minimum uncontrolled miss distance occurs (sec.)

ρ - minimum controlled miss distance (ft.)

t^o - time at which minimum controlled miss distance occurs (sec.)

t^* - maximum permissible period of initial coast

F - thrust vector

SECTION 1

INTRODUCTION

We examine the terminal phase of an interception problem in terms of a minimum time formulation. The problem is based on a simplified model; the target is assumed to be non-maneuvering and moving in a plane under the influence of a uniform gravitational field. The entire interception takes place above the sensible atmosphere. The minimum time interception steering program is a constant angle in inertial space. The simple form of the optimal control is appealing; there may be situations where time optimality is unimportant, yet the form of the control law may prove to be a significant factor in the design of the flight control and guidance systems. The terminal phase of an interception may take place in a very short span of time. For example, only about two seconds elapse between the time a target is acquired and the time a target is passed if the initial range is 20 miles and the relative speed is 50,000 ft/sec. In this study we do not try to map out the regions of accessibility for different interceptor rocket parameters. We do present some typical trajectories so that the nature of the optimal paths can be seen in some specific instances. Our goal in the work was to see how the optimal control could be used in an operational sense, to point out the critical areas from the standpoint of feasibility, and to provide an analysis and a computer program for simulated interceptions. Relative velocities ranging from 5000 ft/sec. to 50,000 feet, and initial (no control) miss distances up to 43,000 feet are used in the illustrative examples.

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The error coefficients, i.e., the partial derivatives of the final miss distance with respect to initial conditions, form an essential part of the study. These coefficients depend on all the initial conditions. The computation of the optimal control depends on the use of measured values of the relative range and velocity between the target and interceptor. These measurements may be accomplished by ground based, or missile based equipment. The control computer uses these measurements of relative position and velocity to calculate the optimal steering program. Errors in measuring position and velocity generate errors in the calculated steering program which in turn cause errors in the path and final miss distance. The interceptor must have a means for guiding the direction of the rocket thrust. Errors in carrying out the commanded steering program also cause errors in the end conditions. The sensitivity of the entire system operation to random errors occuring at various points in the control loop must be determined so that steps may be taken to minimize the expected miss distance, variance of the terminal error, or some other performance criterion. We will not consider the problem of optimizing statistical properties of the end conditions. We will outline a method for estimating the statistics of the terminal errors.

The control for a time optimal interception, based on a deterministic formulation, ignites the rocket as soon as the target is sighted. A control policy based on stochastic considerations might not do this. The initial estimates of relative position and velocity may be poor and might be expected to improve as more observations are made along the coasting path. That is, it may be better to continue coasting towards the target

making measurements and waiting for an opportune time to ignite the rocket. It still remains to specify what is meant by an opportune time. Suppose that once the optimal steering angle is determined the rocket direction is held fixed in space by a stabilization system and further corrections are not allowed; this is an open-loop terminal guidance scheme. The best time to fire is the time when the sensitivity to errors in the initial conditions is least provided that the intercept can still be made. The definition of least sensitivity to errors is open to interpretation, but this will not be pursued in any greater detail at this point. Corrections to the steering angle made during powered flight require a closed-loop terminal guidance system. The sensitivity to errors depends not only on the sensitivity coefficients taken at the initial time but on the integrated effect of errors made during powered flight. Several sample sets of open-loop error coefficients are computed along coasting paths, as well as closed-loop coefficients computed along optimal trajectories. These error coefficients yield the expected terminal errors for both open-loop systems and closed-loop flight control systems for the sample optimal paths. The closed-loop analysis requires knowledge of the noise characteristics of the sensors and other components of the guidance systems.

SECTION 2

FORMULATION OF THE PROBLEM

1. Time Optimal Interception.

The simplified equations of motion are shown below in canonical form

$$\dot{x} = v_x$$

$$\dot{y} = v_y$$

$$\dot{v}_x = \frac{c_o}{m_o/\beta - t} \cos \alpha$$

$$\dot{v}_y = \frac{c_o}{m_o/\beta - t} \sin \alpha$$

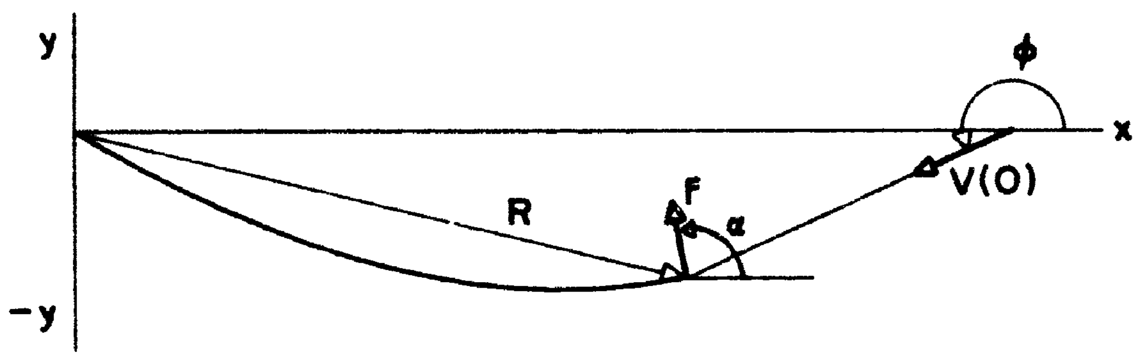

Figure 1. Coordinate System

We want to bring the state variables x and y to zero simultaneously in the least time. There are no specifications placed on the terminal values of v_x and v_y.

In the equations above, c_o is the rocket characteristic exhaust velocity, m_o is the initial system mass, β is the propellant mass flow rate, and α is the direction of the rocket thrust measured from the positive x axis. The coordinate system is shown in Fig. 1. The target

is assumed to be falling freely under a uniform gravitational field. The origin of the coordinate system is placed at the target. The entire interception is assumed to take place above the atmosphere.

The optimal control is easily obtained from the maximum principle (1, 2). Introduce adjoint variables $p_i (i = 0...4)$ and form the Hamiltonian

$$H = \sum_{i=0}^{4} p_i \dot{x}_i = p_1 \dot{x} + p_2 \dot{y} + p_3 \dot{v}_x + p_4 \dot{v}_y + p_o$$

or

$$H = p_1 v_x + p_2 v_y + p_3 \frac{c_o}{m_o/\beta - t} \cos \alpha + p_4 \frac{c_o}{m_o/\beta - t} \sin \alpha + p_o$$

Maximizing H over all α yields the result

$$\text{Tan } \alpha^o = \frac{p_4(t)}{p_3(t)}$$

The adjoint variables p_i satisfy the equations

$$\dot{p}_i = - \frac{\partial H}{\partial x_i}$$

$$p_1(t) \equiv p_2(t) \equiv 1$$

$$p_3(t) = - p_1(0)$$

$$p_4(t) = - p_2(0)$$

Therefore

$$\text{Tan } \alpha^o = \frac{-p_1(0)}{-p_2(0)} = \text{constant}$$

The optimal angle, α^o , and the optimal time t^o for each initial condition can be found by solving a pair of transcendental equations. These equations are obtained from the integration of the equations of motion:

$$x(t^o) = 0 = \frac{c_o m_o}{\beta} \left[\left(1 - \frac{t^o \beta}{m_o}\right) \ln \left(1 - \frac{t^o \beta}{m_o}\right) + \frac{t^o \beta}{m_o} \right] \cos \alpha^o + v_x(0) t^o + x(0)$$

$$y(t^o) = 0 = \frac{c_o m_o}{\beta} \left[\left(1 - \frac{t^o \beta}{m_o}\right) \ln \left(1 - \frac{t^o \beta}{m_o}\right) + \frac{t^o \beta}{m_o} \right] \sin \alpha^o + v_y(0) t^o + y(0)$$

These equations may be solved by squaring, eliminating α^o, and finding t^o. The relations between t^o, α^o and the initial conditions are then used to find α^o.

A graphical solution developed by Faulkner is described in Reference 2. The method used in this study is described in the Appendix.

2. Closed-Loop Systems with Random Noise.

The solution of the equations developed in Part 1 of this section gives the optimal steering angle as a function of the initial conditions. The closed-loop flight control system based on this steering law will have to operate in the presence of noise. We distinguish between the actual state of the system, the measured state, and the best estimate of the state; the control is computed on the basis of the best estimate of the state. If no filter is used, the control is computed directly from the measured state variables. The actual trajectory evolves from the initial state, $x(0)$, in response to the applied control. With no feedback the trajectory $x(t)$ would miss the target by an amount calculable from the equations of motion and the error in control application. The information on measurement errors, disturbances, and control actuation errors is assumed to be statistical, i.e., we may not know in advance what the errors will be but we may know something about the probability density distribution of the errors. Let the difference between the nominal trajectory $x^o(t)$ and the actual trajectory $x(t)$ be called δx i.e.: $x(t) = x^o(t) + \delta x$. The equations governing the motion are:

$$\dot{x}_1(t) = f_1(x, \alpha) \qquad (1)$$

where x is an n-vector and α is a scalar control. We assume, at least for the present, that f is differentiable with respect to x and α. We will also assume that the optimal control $\alpha^o(x)$ can be differentiated everywhere with respect to x. We shall restrict the optimal nominal paths to those which can be embedded in a field and eliminate a large class of problems, e.g., bang-bang minimum time controllers. It is possible to analyze the bang-bang problems separately and they will not be considered here at all. Kalman has studied problems of optimal filtering and control for linear systems with quadratic performance criteria (3,4). Our analysis bears a resemblence to some of that work, but the motivation here is different. We want to estimate the statistical properties of the terminal error, but we are not going to synthesize an optimal filter.

We take up the case of no filtering. We want to look at the difference between the actual path and the nominal path and eliminate the observed variables. The observed values are $\hat{x} = x(\text{actual}) + N = x^o + \delta x + N$ where N is additive noise. The control applied is $\alpha = \alpha^o + \delta\alpha$ where $\delta\alpha = \left.\frac{\partial\alpha^o}{\partial x}\right|_{x^o}(\delta x + N) + \tilde{\alpha}$. The error in applying the control is $\tilde{\alpha}$. The sketch below illustrates the general idea.

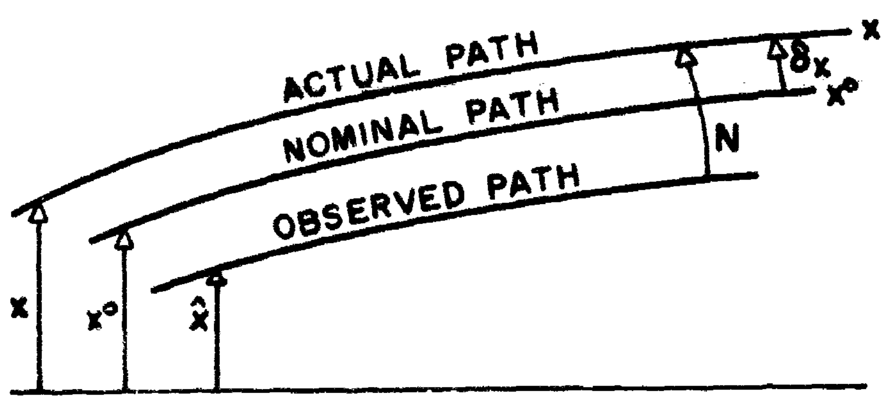

Figure 2. Relation between Actual Path, Nominal Path and Observed Path

All partial derivatives are evaluated along the nominal path. The $|_{x^o}$ will not be used unless there is a possibility of ambiguity. It follows from the equations of motion that

$$\delta\dot{x} = \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial \alpha}\,\delta\alpha \tag{2}$$

Substituting for $\delta\alpha$ we find that

$$\delta\dot{x} = \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial \alpha}\left[\frac{\partial \alpha^o}{\partial x}(\delta x + N) + \tilde{\alpha}\right] \tag{3}$$

Rearranging terms we get:

$$\delta\dot{x} = \left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \alpha}\frac{\partial \alpha^o}{\partial x}\right]\delta x + \frac{\partial f}{\partial \alpha}\frac{\partial \alpha^o}{\partial x} N + \frac{\partial f}{\partial \alpha}\tilde{\alpha} \tag{4}$$

The response of the system to initial errors, measurement errors, and actuation errors is evaluated by solving Equation (4).

The changes in control, $\delta\alpha$, depend linearly on the actual disturbances in the state, i.e., $\delta\alpha = \frac{\partial \alpha^o}{\partial x}\Big|_{x^o}\delta x$. The term $\frac{\partial \alpha^o}{\partial x}$ is just a time-varying gain. If we have already synthesized an optimal controller and $\frac{\partial \alpha^o}{\partial x}$ is available we can examine the behavior of closed-loop optimal systems. The analysis, however, does not depend on having an optimal control. It requires only a nominal path and a linearized feedback equation.

This means that Equation (4) can be used to study any linearized feedback system with time-varying gains. Furthermore, there is the possibility of selecting these time-varying gains to optimize a performance criterion associated with the closed-loop errors. This is the approach Kalman takes in the previously mentioned work on optimal linear feedback systems with quadratic criteria.

Two examples are given to show how this method can be applied.

Example 1. Linear Servomechanism (regulator).

The system is described by the following equation

$$\dot{x} = Ax + Bu$$

where

x is an n-vector

A is $n \times n$ matrix

B is $n \times r$ matrix

u is an r-vector

The commanded control is a linear function of the observed state variables $u = kx = k(x^{o} + \delta x + N)$. The control actually applied, $\hat{u}$, is assumed to be the commanded control, u, modified by additive noise, i.e. $\hat{u} = u + \tilde{u}$, where $\tilde{u}$ is the error in the control. The vector N is the noise added to the state variables. The actual errors in the state variables are obtained by substitution.

$$\delta\dot{x} = \left(A + B\frac{\partial u^{o}}{\partial x}\right)\delta x + B\frac{\partial u^{o}}{\partial x}N + B\tilde{u} \tag{5}$$

a) If $\tilde{u} \equiv 0$, i.e. perfect control, and $N = 0$, no noise, then $\delta x(t)$ is determined by $\delta x(0)$ only.

b) If $\delta x(0) = 0$, then $\delta x(t)$ is determined by the noises N and $\tilde{u}$. The linear case with Gaussian distributions for the random errors can be solved completely by well-known methods.

Example 2. A Time-Optimal Steering Problem.

Suppose we desire to steer a vehicle with constant speed, v, from one location to another in least time. The equations of motion are

$$\dot{x} = v\cos\alpha \tag{6a}$$

$$\dot{y} = -v\sin\alpha \tag{6b}$$

where α is the angle between the velocity vector and the x-axis measured positively counterclockwise.

Suppose the target is a circle of radius a centered at the origin as shown in Figure 3 below.

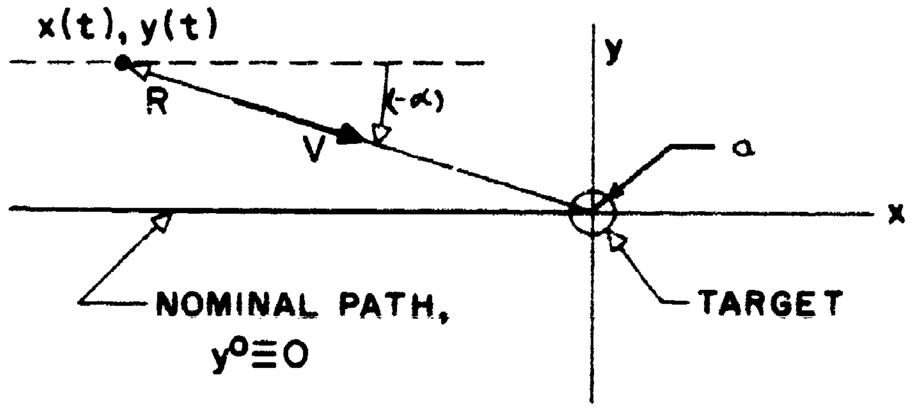

Figure 3. Time Optimal Steering Example

It is easy to show that the time optimal steering angle for this problem directs the velocity vector towards the target at each instant of time. The isochrones are circles centered at the origin. The application of Equation (4) yields

$$\delta\dot{x} = v\left[-\sin\alpha^o\left\{\frac{\partial\alpha^o}{\partial x}(\delta x+N_1) + \frac{\partial\alpha^o}{\partial y}(\delta y+N_2) + \tilde{\alpha}\right\}\right] \quad (7a)$$

$$\delta\dot{y} = v\left[\cos\alpha^o\left\{\frac{\partial\alpha^o}{\partial x}(\delta x+N_2) + \frac{\partial\alpha^o}{\partial y}(\delta y+N_2) + \tilde{\alpha}\right\}\right] \quad (7b)$$

The equation of interest is (7b). We are free to choose the orientation of the coordinate system axes. The nominal initial conditions can be taken as: $x(0) = x_o$, $y(0) = 0$ and the nominal optimal path will be $y^o(t) \equiv 0$.

Along the nominal path $\left.\frac{\partial\alpha^o}{\partial x}\right|_{y^o} = \left.\frac{y}{R^2}\right|_{y^o} = 0$ and $\left.\frac{\partial\alpha^o}{\partial y}\right|_{y^o} = -\frac{\partial}{\partial y}\left[\sin^{-1}\frac{y}{\sqrt{x^2+y^2}}\right]_{y^o} = -\frac{x}{R^2} \sim -\frac{1}{R}$ for $x^2 \gg y^2$ where $R^2=x^2+y^2$.

The equation for $\delta\dot{y}$ becomes

$$\delta\dot{y} = -\frac{v}{R}(\delta y+N_2) + v\tilde{\alpha}$$

or

$$\delta\dot{y} = -\frac{v}{x(0)-vt}\,\delta y - \frac{vN_2}{x(0)-vt} + v\tilde{\alpha}$$

The equation for the closed-loop optimal control has a singularity at the nominal interception time; $t = \frac{x(0)}{v}$. The net effect is to create a feedback gain inversely proportional to R.

If the initial error $\delta y(0)$ is known and if the noise N_2 and actuation error are known to be statistically independent then the statistical behavior of $\delta y(t)$ can be found from the Fokker-Planck equation (5). In the case of Gaussian noise, the solutions are easily obtained by more direct methods. For example, the solution of the differential equation can be written explicitly as

$$\delta y(t)=\delta y(0)\left[\frac{x(0)-vt}{x(0)}\right]+\left(\frac{x(0)-vt}{x(0)}\right)\int_0^t \frac{-x(0)v}{(x(0)-v\tau)^2}\,N_2(\tau)d\tau$$

$$+\left(\frac{x(0)-vt}{x(0)}\right)\int_0^t \frac{x(0)v}{x(0)-v\tau}\,\tilde{\alpha}(\tau)d\tau$$

Suppose $\frac{\partial \alpha}{\partial y} = K(t)$ where $K(t)$ is prescribed in advance instead of the optimal feedback. The equations then are

$$\delta\dot{y} = vK(t)\ (\delta y+N_2) + v\tilde{\alpha}$$

If $K(t)$ is constant, then

$$\delta\dot{y} = vK(\delta y+N_2) + v\tilde{\alpha}$$

and the steering problem now is a special case of the servomechanism problem used in Example 1 with $n = 1$.

This first order differential equation represents a linear transformation of Gaussian processes. The random function $\delta y(t)$ is the sum of two linearly transformed Gaussian processes and has a Gaussian distribution.

Suppose, for example, that $N_2 \equiv 0$. Then, returning to Example 2,

$$\delta y(t) = \delta y(0)\left[\frac{x(0) - vt}{x(0)}\right] + \frac{x(0) - vt}{x(0)}\int_0^t \frac{x(0)v}{x(0) - v\tau}\,\tilde{\alpha}\,d\tau$$

$$E(\delta y(t)) = E(\delta y(0))\frac{(x(0) - vt)}{x(0)} + \frac{x(0) - vt}{x(0)}\int_0^t \frac{x(0)v}{x(0) - v\tau}E(\tilde{\alpha})\,d\tau$$

and

$$\sigma^2_{\delta y} = E(\delta y(t) - \delta\bar{y})^2 = \left[\frac{x(0) - vt}{x(0)}\right]^2 \sigma^2_{\delta y(0)}$$

$$+ \left[\frac{x(0) - vt}{x(0)}\right]^2 \int_0^t\int_0^t \frac{v^2 x^2(0)}{(x(0) - v\tau_1)(x(0) - v\tau_2)} k_{\tilde{\alpha}}(\tau_1, \tau_2)\,d\tau_1 d\tau_2$$

where $k_{\tilde{\alpha}}(\tau_1, \tau_2) \equiv E\left\{(\tilde{\alpha}(\tau_1) - \bar{\alpha})(\tilde{\alpha}(\tau_2) - \bar{\alpha})\right\}$ is the autocorrelation function for $\tilde{\alpha}$ (6). The same result can be obtained via the weighting function-power spectral density approach (7).

The analysis discussed in the preceding paragraph can be applied in a relatively simple way to improve non-optimal feedback about an optimal path. Suppose that we use a constant gain feedback ($K < 0$) based on the displacement from the nominal path. If the actuator errors are negligible compared to the measurement errors, we may write: $\delta\dot{y} = vK\delta y + vKN$.

Suppose also that the initial errors and measurement noise have Gaussian distributions with zero means. Let the variance of the measurement noise be q, and the noise be "white" such that $k_N(\tau_1, \tau_2) = q\delta(\tau_1 - \tau_2)$ and δ is the Dirac Delta function; the variance of the initial errors be $\sigma^2(0)$. The variance at the terminal

time t^{o} is given by

$$\sigma^2_{\delta y(t^o)} = \sigma^2_{\delta y(0)} e^{2vKt^o} - \frac{qvK}{2}(1 - e^{2vKt^o})$$

and $\delta y(t^o)$ has a Gaussian distribution with zero mean and variance, $\sigma^2_{\delta y(t^o)}$. It is possible to find a value of gain K that minimizes $\sigma^2_{\delta y(t^o)}$ (note that $\frac{\partial}{\partial K}\sigma^2_{\delta y(t^o)}$ < 0 at $K = 0$, and $\sigma^2_{\delta y(t^o)} \longrightarrow +\infty$ as $K \longrightarrow +\infty$). We will not carry these simple examples any further. They serve the purpose of showing that the error coefficients are desirable objects to obtain. They play an important role in evaluating the behavior of a control system in the presence of random noise. In the next section, we will examine the dependence of the error coefficients on initial conditions.

SECTION 3

COMPUTATIONAL STUDIES

The discussion in Section 2 was directed at the analytical formulation of the interception problem. In this section, we examine the sensitivities of the interception paths to errors. Figure 4 shows a typical interception configuration. The parameters of interest are the initial miss distance, $\rho_o = R(0) \sin \phi$, the initial range $R(0)$, and the initial velocity $V(0)$. The thrust level is not adjustable; the optimal steering angle can be found by the method described in the appendix.

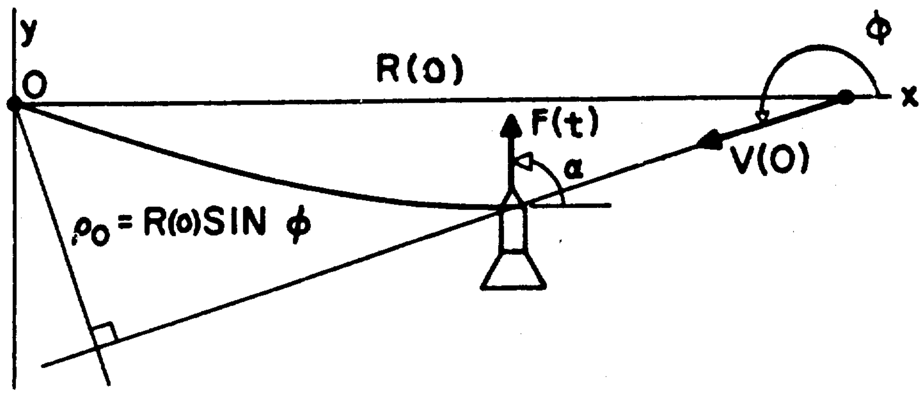

Figure 4. Typical Intercept Configuration

Table I contains a partial listing of the cases studied. The miss distances or distances of closest approach, ρ, are given for five values of initial range and four values of initial velocity. The results are grouped according to the initial range; each group is ordered according to the initial miss distance ρ_o.

Two rockets were used in this study and their characteristics are listed below. These values are typical of the kind of rockets being considered for use in missile and satellite interception studies.

Rocket #1

c_o = 10,000 ft/sec - effective exhaust velocity

m_f = .15 m_o, β/m_o = .1/sec, t_b = 8.5 sec

a (0) = 30 g (approx.)

a (t_b) = 200 g (approx.)

ΔV_{ideal} = 18,900 ft/sec

Rocket #2

c_o = 10,000 ft/sec

m_f = .15 m_o, β/m_o = .05/sec, t_b = 17.0 sec

a (0) = 15 g (approx.)

a (t_b) = 100 g (approx.)

ΔV_{ideal} = 18,900 ft/sec

These rockets have the same mass ratio (m_f/m_o) and the same ΔV_{ideal}. They differ in the burning rate of the propellant and in the acceleration time history.

Two sets of trajectories are shown in Figures 6 and 7 . (Note that the x-y scales are not the same.) Each set of figures shows the effect of initial speed on the interception paths; Figure 6 is for Rocket #1, Figure 7, for Rocket #2. It may help to refer back to Figure 1 where the coordinate system is defined.

The trajectories shown in Figures 6 and 7 represent simulated interceptions with the rocket ignited at the point of acquisition of the target.

Imagine a situation where there is a possibility of delaying the ignition of the rocket to improve the relative position and velocity measurements.

Figure 8 shows the family of interception paths for Rocket #1 emanating from the coasting path starting at R(O). The target is sighted at a range of 500,000 feet and the interceptor approaches the target on the coasting path at a speed of 10,000 ft/sec. The miss distance ρ_0 in the absence of terminal control is 4,363 feet.

Each path in this family is obtained by solving for the optimal steering angle and computing the optimal trajectory. The optimal steering angles, α^o, are shown in Figure 8 next to the starting points. The optimal steering program to the target from any starting point is constant and is measured from the positive x-axis as shown in Figure 1. This angle depends on the initial conditions and it changes as the starting points vary along the initial coasting path. The optimal steering angle changes by about only 7^o as the initial range decreases from 500,000 feet to about 100,000 feet. The rate of change of steering angle increases sharply thereafter. The effect of a time delay in igniting the rocket can be determined by an examination of the rate of change of α^o along the coasting path.

Open-loop miss sensitivities have been found for each of the starting points shown in the figure. These partial derivatives represent the sensitivity of terminal errors to errors in initial measurements and to errors in steering. These are shown next in Figure 9. The sensitivity of the error in $y(T)$ at the distance of closest approach to errors in α^o is shown in Figure 9. The sensitivity of $y(T)$ to errors in α^o decreases continually as the target is approached until the range is very small. A sharp increase in steering angle sensitivity with decreasing range is shown for the other initial conditions in

the remaining figures of this set. This is not unexpected; the α^o listings on Fig. 8 show a rapid change in the initial steering angle as the range approaches 40,000 ft. The paths change from head-on interceptions to beam approach interceptions in this region as the interceptor coasts by the target. The initial relative velocity of 10,000 ft/sec is low enough to permit the interceptor to reverse its direction of motion. Two paths shown in Fig.8 pass the target and reverse direction. The discontinuous changes in the error sensitivities (shown in Fig. 9) occur as the interceptor passes a beam of the target.

It is possible to compute some of the partial derivatives directly from the integrals of the equations of motion, e.g.,

$$\frac{\partial x(t^o)}{\partial \alpha^o} \equiv \left.\frac{\partial x(t^o)}{\partial \alpha}\right|_{\alpha=\alpha^o} = \frac{c_o m_o}{\beta}\left[\left(1-\frac{t^o\beta}{m_o}\right)\ell n\left(1-\frac{t^o\beta}{m_o}\right)+\frac{t^o\beta}{m_o}\right](-\sin\alpha^o)$$

$$\frac{\partial y(t^o)}{\partial \alpha^o} \equiv \left.\frac{\partial y(t^o)}{\partial \alpha}\right|_{\alpha=\alpha^o} = \frac{c_o m_o}{\beta}\left[\left(1-\frac{t^o\beta}{m_o}\right)\ell n\left(1-\frac{t^o\beta}{m_o}\right)+\frac{t^o\beta}{m_o}\right]\cos\alpha^o$$

$$\frac{\partial y(t^o)}{\partial y(0)} = 1$$

These derivatives are evaluated at the nominal optimal time, t^o. The derivatives of the form $\frac{\partial \alpha^o}{\partial x(0)}$, $\frac{\partial \alpha^o}{\partial \dot{x}(0)}$, etc., are more difficult to obtain in closed form as the optimal angle is given implicitly.

Trajectories with initial relative speeds of 25,000 ft/sec and 50,000 ft/sec are shown in Figures 10 and 12 with Rocket #1 and in Figures 14 and 16 with Rocket #2. Error coefficients for these cases are given in Figures 11, 13, 15, and 17.

The principal error coefficients

$$\frac{\partial y(T)}{\partial x(0)}, \frac{\partial y(T)}{\partial \dot{x}(0)}, \frac{\partial y(T)}{\partial \dot{y}(0)}$$

are presented in Figures 18 and 19 for the trajectories of Figures 10, 12, 14, and 16. These are evaluated at T, the time of closest approach to the target, with perturbed thrust direction angle resulting from perturbations in the initial conditions. The error coefficients

$$\frac{\partial y(T)}{\partial x(0)}, \frac{\partial y(T)}{\partial \dot{x}(0)}$$

increase in absolute magnitude as the initial range decreases; the coefficient $\frac{\partial y(T)}{\partial \dot{y}(0)}$ decreases in absolute magnitude as the range decreases. The location of the best place to ignite the rocket from the standpoint of open-loop terminal error dispersion can be found by using the noise characteristics of the sensors.

The analysis of feedback systems requires the partial derivatives

$$\frac{\partial \alpha^o}{\partial x}, \frac{\partial \alpha^o}{\partial \dot{x}}, \frac{\partial \alpha^o}{\partial y}, \text{ and } \frac{\partial \alpha^o}{\partial \dot{y}}$$

along the optimal path. The latter three derivatives are presented in Figures 20 and 21 for the sample cases discussed earlier. The derivative $\frac{\partial \alpha^o}{\partial x}$ is very small except at the very end of the flight and is not plotted here. The derivatives are plotted as functions of R, range. The differential equations require these as functions of time and the computer program provides both forms. The derivatives generated by the computer program can be used as indicated in the following paragraphs.

If the equations for error propagation in an optimally-controlled system are expanded, we obtain

$$\delta\dot{x} = \delta v_x$$

$$\delta\dot{v}_x = -a(t)\sin\alpha^o\left[\sum_{i=1}^{4}\frac{\partial\alpha^o}{\partial x_i}\delta x_i + \sum_{i=1}^{4}\frac{\partial\alpha^o}{\partial x_i}N_i\right] - \left[a(t)\sin\alpha^o\right]\tilde{\alpha}$$

$$\delta\dot{y} = \delta v_y$$

$$\delta\dot{v}_y = a(t)\cos\alpha^o\left[\sum_{i=1}^{4}\frac{\partial\alpha^o}{\partial x_i}\delta x_i + \sum_{i=1}^{4}\frac{\partial\alpha^o}{\partial x_i}N_i\right] + \left[a(t)\cos\alpha^o\right]\tilde{\alpha}$$

where $a(t) = \dfrac{c_o}{m_o/\beta - t}$, $x_1 = \delta x$, $x_2 = \delta v_x$, $x_3 = \delta y$, $x_4 = \delta v_y$, N_i = measurement errors, and $\tilde{\alpha}$ is the error in applying the commanded control.

The quantity $a(t)$ is known from the rocket specifications. The optimal angle, α^o, is a constant along the nominal path.

The specification of the statistical character of the noise N_i and actuation errors $\tilde{\alpha}(t)$ provides the information needed to evaluate the statistical properties of the terminal errors. The function $a(t)$ is a rapidly increasing function of time. The derivatives $\frac{\partial\alpha^o}{\partial\dot{x}}$, $\frac{\partial\alpha^o}{\partial y}$, $\frac{\partial\alpha^o}{\partial\dot{y}}$, are also increasing functions of time. The optimal steering angle becomes very sensitive to measurement errors. This may be thought of as a rapid increase in a feedback gain near the end of powered flight.

A time-varying feedback control gain that takes into account the variation in measurement error statistics along the flight path may prove to be better than the closed-loop time optimal controller from the standpoint of error sensitivity.

The error propagation is governed by a set of inhomogeneous time-varying linear differential equations. The forcing functions acting on these equations are the measurement errors $N_i(t)$ and the actuation errors $\tilde{\alpha}(t)$.

Large measurement errors during powered flight may cancel the benefits derived from feedback of the estimated position and velocity errors. If there were no measurement errors, i.e., $N_1 \equiv 0$, then the terminal errors due to $\tilde{\alpha}(t)$ can be reduced by the use of feedback of position and velocity signals. The presence of measurement noise, N_1, makes necessary a detailed analysis of closed-loop guidance. It seems fairly clear that if the noise level is high compared to the initial errors, $\delta x_1(0)$, it may be desirable to run the system open-loop. The terms closed-loop and open-loop apply to the use of commanded changes in the steering angle due to errors in position and velocity along the nominal path. The optimal steering angle in either case is to be controlled or stabilized in direction by some other attitude control system.

We will illustrate the use of results of the computer program by a simple single variable open loop example. Suppose that the measurements of the single variable y are corrupted by additive noise, the variance of the measurement noise depends linearly on range, and the commanded control has no errors i.e. $\tilde{\alpha} \equiv 0$. Suppose also that one of the graphs in Figure 18 represents the sensitivity of terminal errors to errors in initial determination of position and that this curve can be satisfactorily approximated by an exponential function. It is relatively simple to find the best range for initiating the control to give the smallest variance in the terminal errors. The best location occurs either at an interior point or at one of the extreme ends of the interval beginning at the point of target acquisition and ending at the last point where an interception can be initiated. (This interval may not be closed.) We assume that the terminal error depends linearly on $\delta y(0)$ for small initial errors so that

$$\sigma^2_{y(T)}(x) = \left[\left.\frac{\partial y(T)}{\partial y(0)}\right|_{y(0) = \bar{y}(0)}\right]^2 \sigma^2_{y(0)}(x)$$

An interior minimum is sought by differentiating the expression (where y is used in place of $y(0)$)

$$\sigma^2_y(x) \cdot \left[\frac{\partial y(T)}{\partial y(0)}\right]^2$$

with respect to range (x). In this example, let

$$\sigma^2_y(x) = b + cx \qquad b,\ c > 0, \qquad x \geq 0$$

and

$$\left[\frac{\partial y(T)}{\partial y(0)}\right]^2 = a^2 e^{-2\lambda x} \qquad \lambda < 0$$

We must look at the zero of

$$\frac{d}{dx}\left[(a^2 e^{-2\lambda x}) \cdot (b + cx)\right]$$

and see if it lies in the interval of interest and test to see if we have a maximum or a minimum. Even this simple example provides some interesting results. The derivative of the terminal variance with range is

$$-a^2 e^{-2\lambda x}\left[2\lambda(b + cx) - c\right]$$

The vanishing of the derivative corresponds to a <u>maximum</u> sensitivity; the minimum sensitivity occurs at one of the endpoints. The minimum location depends strongly on the parameter b. For $b = 0$, the derivative vanishes at $x = \frac{1}{2\lambda}$. The interior extrema do not depend on the parameter a at all.

This procedure can be applied to multivariable problems in a straightforward way using co-variance matrices instead of the scalar variance used in the example.

SECTION 4

PROPORTIONAL CONTROLLERS

Interceptions with proportional control systems were simulated on a digital computer. Three modes of control were studied:

Mode A - The thrust is directed along the line-of-sight towards the target. The possibility of using a constant lead angle is included.

Mode B - The thrust is directed off the line-of-sight by an angle proportional to $\frac{\vec{V} \times \vec{r}}{|r|}$ in the sense required to reduce the rate of rotation of the line-of-sight.

Mode C - The thrust is applied normal to the line-of-sight in the direction needed to reduce the rate of rotation of the line-of-sight. If the interceptor velocity becomes aligned with line-of-sight, there is an option either to stop thrusting or to thrust along the line-of-sight.

By putting limits on the steering angle and raising the gain, we can also obtain thrusting perpendicular to the line-of-sight from Mode B. Firing perpendicular to the line-of-sight worked quite well.

We did not explore all the possible combinations of initial conditions and control gains. The reachable sets for the optimal steering were not mapped in detail. It is not difficult to determine on the basis of physical considerations which of the possible control schemes are most desirable. The control must direct the relative velocity vector towards the origin before the rocket burns out. If the time of flight in the terminal phase is shorter than the rocket's burning time and if the transverse velocity component is large compared with ΔV available for the rocket, then it is desirable to fire

slightly backwards. This lengthens the time available for reducing the transverse component. In effect, this is a trade between interception time and terminal error. Firing along the line-of-sight is undesirable in this situation. On the other hand, if the transverse velocity is low, the initial miss distance is small, and the time of flight is long, then the time optimal steering angle is very close to the line-of-sight. This can be seen in Figure 23.

Figure 22 shows five trajectories starting from identical initial conditions. Path number 1 is the time optimal path; thrusting perpendicular to the line-of-sight, Mode C, gives path number 2. Mode B produces path 3 with a miss distance of 24,000 feet. The result of steering along the line-of-sight, Mode A, is the path marked 4, with a miss distance of 91,000 feet. The steering angles used are given in Figure 23.

The gains used in these runs were selected a priori and were not optimized with respect to the miss distances. The optimal control system employs a computer to solve a transcendental equation; the application of this computer to the problem of gain and lead angle selection would undoubtedly improve the performance of the proportional controllers.

The optimal controller nevertheless possesses advantages which are not matched by the proportional controllers. The optimal-control computer will minimize the miss distance instead of interception time if a hit cannot be scored. The computation of the optimal control also takes into account the terminal coasting phase after the propellant is exhausted. Furthermore, the optimal control produces hits for initial conditions which lie on the boundary of the region of attainability whereas the proportional controllers need not do this.

SECTION 5

SUMMARY AND CONCLUSIONS

A planar time optimal formulation is used to study the terminal phase of an interception problem. The optimal steering program for interception in a uniform gravitational field in the absence of aerodynamic drag requires the thrust direction to be held constant in inertial space during the powered portion of flight. The optimal steering angle is a function of the relative initial position and relative velocity. We have examined the dependence of the optimal steering angle on the initial conditions for initial velocities ranging from 5,000 ft/sec to 50,000 ft/sec. The results are presented in graphical form showing: (1) the range dependence of the terminal error sensitivities to error in initial position and relative velocity measurements taken along an initial rectilinear coasting path, and (2) the variation in these terminal error sensitivities along an optimal interception path.

Simulated interceptions were made using proportional controllers. Three modes of control were tried; thrusting perpendicular to the line-of-sight gave good results in the cases studied. The optimal controller is predictive in nature and takes into account the possibility of a terminal coasting phase after the propellant is exhausted. The computer also predicts whether or not a hit is possible and will minimize the miss distance if an interception is not possible. Further work is needed to improve the performance of the proportional controllers, e.g., selecting the gain and lead angle as functions of range, velocity or combinations of other variables before a definitive comparison can be made with optimal controllers.

One of the results of our work is a flexible digital computer program with many options, which can simulate optimal interceptions, generate the error sensitivities along initial coasting paths and along optimal paths and make terminal error dispersions studies. A listing of the program and a set of cards are available at ARAP.

The study of error sensitivities raised several important questions which are not yet settled. The computer program, as it now stands, is able to simulate closed-loop interceptions. Several check problems were run but systematic studies were not made because of the dearth of unclassified realistic data on radar and sensor noise and missile-borne guidance and attitude control systems. Future research programs in this area could take advantage of this computer routine to investigate the behavior of optimally-controlled hypervelocity interceptions, with and without relative position and velocity feedback during the burning portion of the flight. This simulation should include measurement errors and actuation errors based on experience with actual equipment tests.

SECTION 6

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TABLE 1

SUMMARY OF INTERCEPTION RUNS WITH OPTIMAL CONTROL

Rocket#1

Initial Range R(0)(ft)	Miss Distance without control ρ_0 (ft)	Miss Distance with optimal control ρ (ft)	Initial Velocity (ft/sec) V(0)	Optimal Steering Angle α^o
50,000	873	.000	10,000	175.77
"	"	.000	15,000	171.03
"	"	.000	25,000	155.01
"	"	355.0	50,000	89.79
"	1,309	.000	10,000	173.66
"	"	.000	15,000	166.50
"	"	.000	25,000	140.94
"	"	791.0	50,000	90.29
"	1,745	.000	10,000	171.54
"	"	.000	15,000	161.91
"	"	.000	25,000	123.70
"	"	1,228.0	50,000	90.79
"	4,358	.000	10,000	158.75
"	"	.000	15,000	131.47
"	"	2,209.0	25,000	89.86
"	"	3,843.0	50,000	93.80
"	8,682	.000	10,000	136.70
"	"	.000	15,000	13.08
"	"	6,567.0	25,000	94.88
"	"	8,179.0	50,000	98.81
125,000	2,182	.000	10,000	178.29
"	"	.000	15,000	176.53
"	"	.000	25,000	170.72
"	"	.000	50,000	139.36
"	3,272	.000	10,000	177.43
"	"	.000	15,000	174.79
"	"	.000	25,000	166.03
"	"	.000	50,000	105.63
"	4,362	.000	10,000	176.58
"	"	.000	15,000	173.05
"	"	.000	25,000	161.28
"	"	936.0	50,000	88.70
"	10,894	.000	10,000	171.48
"	"	.000	15,000	162.64
"	"	.000	25,000	129.56
"	"	7,480.0	50,000	91.70
"	21,706	.000	10,000	163.17
"	"	.000	15,000	145.35
"	"	6,005.0	25,000	89.98
"	"	18,355.0	50,000	96.72

TABLE 1 (Con't.)

Initial Range R(0)(ft)	Miss Distance without control ρ_0 (ft)	Miss Distance with optimal control ρ (ft)	Initial Velocity (ft/sec) V(0)	Optimal Steering Angle α^o
200,000	3,490	.000	10,000	178.89
"	"	.000	15,000	177.94
"	"	.000	25,000	174.56
"	"	.000	50,000	156.96
"	5,235	.000	10,000	178.34
"	"	.000	15,000	176.92
"	"	.000	25,000	171.83
"	"	.000	50,000	144.28
"	6,980	.000	10,000	177.78
"	"	.000	15,000	175.89
"	"	.000	25,000	169.10
"	"	.000	50,000	129.52
"	17,431	.000	10,000	174.47
"	"	.000	15,000	169.76
"	"	.000	25,000	152.47
"	"	8,067.0	50,000	89.10
"	34,730	.000	10,000	169.03
"	"	.000	15,000	159.78
"	"	.000	25,000	122.65
"	"	25,508.0	50,000	94.12
250,000	4,363	.000	10,000	179.04
"	"	.000	15,000	178.32
"	"	.000	25,000	175.86
"	"	.000	50,000	162.31
"	6,544	.000	10,000	178.56
"	"	.000	15,000	177.48
"	"	.000	25,000	173.80
"	"	.000	50,000	153.01
"	8,725	.000	10,000	178.08
"	"	.000	15,000	176.65
"	"	.000	25,000	171.73
"	"	.000	50,000	143.03
"	21,789	.000	10,000	175.21
"	"	.000	15,000	171.64
"	"	.000	25,000	159.32
"	"	6,366.0	50,000	86.93
"	43,412	.000	10,000	170.49
"	"	.000	15,000	163.41
"	"	.000	25,000	138.76
"	"	28,175.0	50,000	91.92
500,000	8,726	.000	10,000	179.28
"	"	.000	15,000	178.85
"	"	.000	25,000	177.80
"	"	.000	50,000	172.88
"	13,088	.000	10,000	178.93
"	"	.000	15,000	178.28
"	"	.000	25,000	176.70
"	"	.000	50,000	169.30

TABLE 1 (Con't.)

Initial Range R(O)(ft)	Miss Distance without control ρ_O (ft)	Miss Distance with optimal control ρ (ft)	Initial Velocity (ft/sec) V(O)	Optimal Steering Angle α^O
500,000	17,450	.000	10,000	178.57
"	"	.000	15,000	177.71
"	"	.000	25,000	175.60
"	"	.000	50,000	165.71
"	43,578	.000	10,000	176.42
"	"	.000	15,000	174.28
"	"	.000	25,000	168.99
"	"	.000	50,000	143.41
"	86,824	.000	10,000	172.87
"	"	.000	15,000	168.60
"	"	.000	25,000	157.96
"	"	.000	50,000	89.24

Rocket#2

50,000	873	.000	10,000	171.71
"	"	.000	15,000	161.77
"	"	.000	25,000	121.54
"	"	618.0	50,000	90.41
"	1,309	.000	10,000	167.53
"	"	.000	15,000	152.13
"	"	273.0	25,000	89.08
"	"	1,055.0	50,000	90.91
"	1,745	.000	10,000	163.31
"	"	.000	15,000	141.71
"	"	710.0	25,000	89.58
"	"	1,491.0	50,000	90.91
"	4,358	.000	10,000	135.85
"	"	.000	15,000	5.45
"	"	3,327.0	25,000	92.58
"	"	4,105.0	50,000	94.41
"	8,682	.000	10,000	13.83
"	"	.000	15,000	10.89
"	"	7,672.0	25,000	97.60
"	"	8,435.0	50,000	99.42
125,000	2,182	.000	10,000	176.60
"	"	.000	15,000	172.85
"	"	.000	25,000	160.39
"	"	549.0	50,000	89.47
"	3,272	.000	10,000	174.90
"	"	.000	15,000	169.26
"	"	.000	25,000	149.91
"	"	1,640.0	50,000	89.97
"	4,362	.000	10,000	173.19
"	"	.000	15,000	165.65
"	"	.000	25,000	138.38
"	"	2,731.0	50,000	90.47

TABLE 1 (Con't.)

Initial Range $R(0)$ (ft)	Miss Distance without control ρ_0 (ft)	Miss Distance with optimal control ρ (ft)	Initial Velocity (ft/sec) $V(0)$	Optimal Steering Angle α^0
125,000	10,894	.000	10,000	162.97
"	"	.000	15,000	142.88
"	"	4,029.0	25,000	88.34
"	"	9,271.0	50,000	93.47
"	21,706	.000	10,000	145.89
"	"	.000	15,000	15.98
"	"	14,932.0	25,000	93.35
"	"	20,117.0	50,000	98.49
200,000	3,490	.000	10,000	177.85
"	"	.000	15,000	175.60
"	"	.000	25,000	168.17
"	"	.000	50,000	124.64
"	5,235	.000	10,000	176.78
"	"	.000	15,000	173.40
"	"	.000	25,000	162.14
"	"	936.0	50,000	88.94
"	6,980	.000	10,000	175.71
"	"	.000	15,000	171.20
"	"	.000	25,000	155.95
"	"	2,680.0	50,000	89.44
"	17,431	.000	10,000	169.31
"	"	.000	15,000	157.88
"	"	.000	25,000	106.85
"	"	13,151.0	50,000	92.44
"	34,730	.000	10,000	158.86
"	"	.000	15,000	134.95
"	"	15,904.0	25,000	87.86
"	"	30,533.0	50,000	97.46
250,000	4,363	.000	10,000	178.29
"	"	.000	15,000	176.53
"	"	.000	25,000	170.72
"	"	.000	50,000	139.36
"	6,544	.000	10,000	177.43
"	"	.000	15,000	174.79
"	"	.000	25,000	166.03
"	"	.000	50,000	105.63
"	8,725	.000	10,000	176.58
"	"	.000	15,000	173.05
"	"	.000	25,000	161.28
"	"	1,871.0	50,000	88.70
"	21,789	.000	10,000	171.48
"	"	.000	15,000	162.64
"	"	.000	25,000	129.56
"	"	14,960.0	50,000	91.70
"	43,412	.000	10,000	163.17
"	"	.000	15,000	145.35
"	"	12,011.0	25,000	82.98
"	"	36,711.0	50,000	96.72

TABLE 1 (Con't.)

Initial Range $R(0)$ (ft)	Miss Distance without control ρ_0 (ft)	Miss Distance with optimal control ρ (ft)	Initial Velocity (ft/sec) $V(0)$	Optimal Steering Angle α^0
500,000	8,726	.000	10,000	179.04
"	"	.000	15,000	178.32
"	"	.000	25,000	175.86
"	"	.000	50,000	162.31
"	13,088	.000	10,000	178.56
"	"	.000	15,000	177.48
"	"	.000	25,000	173.80
"	"	.000	50,000	153.01
"	17,450	.000	10,000	178.08
"	"	.000	15,000	176.65
" "	"	.000	25,000	171.73
"	"	.000	50,000	143.03
"	43,578	.000	10,000	175.21
"	"	.000	15,000	171.64
"	"	.000	25,000	159.32
"	"	12,732.0	50,000	86.93
"	86,824	.000	10,000	170.49
"	"	.000	15,000	163.41
"	"	.000	25,000	138.76
"	"	56,349.0	50,000	91.92

APPENDIX

INTERCEPT LOGIC AND EQUATIONS

To simplify the bookkeeping in the program we make some changes in notation. Let $x = x_1$, $y = x_2$, $v_x = x_3$, $v_y = x_4$.

The equations of motion of the intercept vehicle with origin at the target are:

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = a(t) \cos \alpha(t)$$

$$\dot{x}_4 = a(t) \sin \alpha(t)$$

where $a(t) = \dfrac{c_o}{m_o/\beta - t} = \dfrac{c_o}{\tau - t}$

$$\tau = \frac{m_o}{\beta}$$

The thrust angle program, $\alpha(t)$, is required to minimize the time to intercept $(x_1 = x_2 = 0)$ or, if that is impossible, to minimize the distance to the target $(\min \sqrt{x_1^2 + x_2^2})$. The application of Pontryagin's Maximum Principle to the problem yields the result that $\alpha(t)$ is equal to some constant α^o . The equations above can then be integrated to yield:

$$x_1(t) = A(t) \cos \alpha^o + \quad x_3(0)t + x_1(0)$$

$$x_2(t) = A(t) \sin \alpha^o + \quad x_4(0)t + x_2(0)$$

$$x_3(t) = \dot{A}(t) \cos \alpha^o + x_3(0)$$

$$x_4(t) = \dot{A}(t) \sin \alpha^o + x_4(0)$$

where

$$A(t) = \begin{cases} B(t) & t \leq t_b \\ B(t_b) + \dot{B}(t_b)(t - t_b), & t \geq t_b \end{cases}$$

$$\dot{A}(t) = \begin{cases} \dot{B}(t), & t \leq t_b \\ \dot{B}(t_b), & t \geq t_b \end{cases}$$

$$\ddot{A}(t) = \begin{cases} \ddot{B}(t), & t < t_b \\ 0, & t > t_b \end{cases}$$

$$\dddot{A}(t) = \begin{cases} \dddot{B}(t), & t < t_b \\ 0, & t > t_b \end{cases}$$

and

$$B(t) = c_o\tau\left[(1 - \frac{t}{\tau})\ell n(1 - \frac{t}{\tau}) + \frac{t}{\tau}\right]$$

$$\dot{B}(t) = -c_o \ \ell n \ (1 - \frac{t}{\tau})$$

$$\ddot{B}(t) \equiv a(t) = \frac{c_o}{\tau - t}$$

$$\dddot{B}(t) = \frac{c_o}{(\tau - t)^2}$$

and t_b denotes the instant at which the rocket engine stops burning.

The problem is now one of determining t^o and α^o such that $x_1(t^o) = x_2(t^o) = 0$ or, if that is impossible, such that $\sqrt{x_1^2(t^o) + x_2^2(t^o)}$ is minimized, t^o being the minimum time to intercept or, if that is impossible, the time to minimum distance from the target. If the coordinate system is changed so that the intercept vehicle is initially at rest at the origin and the target is travelling on a straight line at constant velocity, then $A(t)$ represents the distance of the intercept vehicle from the new origin at time t. The equations of motion of the target vehicle in the new coordinates become:

$$\dot{y}_1 = y_3$$
$$\dot{y}_2 = y_4$$
$$\dot{y}_3 = 0$$
$$\dot{y}_4 = 0$$

or

$$\begin{aligned} y_1(t) &= x_1(0) + x_3(0)t \\ y_2(t) &= x_2(0) + x_4(0)t \\ y_3(t) &= x_3(0) \\ y_4(t) &= x_4(0) \end{aligned}$$

The distance of the target vehicle from the origin in the new coordinate system is defined as $E(t)$, or (Fig. 5a)

$$E(t) = \sqrt{y_1^2(t) + y_2^2(t)}$$

which yields the derivatives to be used later

$$\begin{aligned} \dot{E}(t) &= \frac{1}{E}\Big[y_1(t)y_3(t) + y_2(t)y_4(t)\Big] \\ \ddot{E}(t) &= \frac{1}{E^3}\Big[y_1(t)x_4(0) - y_2(t)x_3(0)\Big]^2 \\ &= \frac{1}{E^3}\Big[x_1(0)x_4(0) - x_2(0)x_3(0)\Big]^2 \\ \dddot{E}(t) &= -\frac{3}{E^5}\Big[y_1(t)y_3(t)+y_2(t)y_4(t)\Big]\Big[x_1(0)x_4(0)-x_2(0)x_3(0)\Big]^2 \end{aligned}$$

At time t the intercept vehicle can reach any point on the circumference of a circle of radius $A(t)$ by suitable choice of α. Hence, the original problem is equivalent to one of determining the smallest t^o such that $A(t^o) = E(t^o)$ or, if that is impossible, the t^o such that $|A(t^o) - E(t^o)|$ is a minimum. The angle α^o can be determined from

$$\begin{aligned} y_1(t^o) &= E(t^o) \cos (\alpha^o - \pi) \\ y_2(t^o) &= E(t^o) \sin (\alpha^o - \pi) \end{aligned}$$

Clearly, whenever $|A(t^o) - E(t^o)|$ is a minimum, $[A(t^o) - E(t^o)]^2$ is a minimum and vice versa. Since the latter function is more tractable analytically, it will henceforth be used to determine t^o. Formally setting the derivative of $[A(t) - E(t)]^2$ equal to zero,

$$\frac{d}{dt}[A(t) - E(t)]^2 \overset{set}{=} 0 = 2[A(t) - E(t)][\dot{A}(t) - \dot{E}(t)]$$

yields a necessary condition for $[A(t) - E(t)]^2$ to be a

minimum; that is, $A(t) = E(t)$, or $\dot{A}(t) = \dot{E}(t)$. As a step in determining the possible roots of these equations, the quantities t_0, t_1, t_2, t_3 are defined as the roots (if they exist) of the equations

$$\dot{E}(t_o) = 0$$

or

$$t_o \triangleq -[x_1(0)x_3(0) + x_2(0)x_4(0)]/[x_3^2(0) + x_4^2(0)]$$

$$\ddot{A}(t_1) - \ddot{E}(t_1) = 0$$

$$\left.\begin{array}{l} \dot{A}(t_2) - \dot{E}(t_2) = 0 \\ \dot{A}(t_3) - \dot{E}(t_3) = 0 \end{array}\right\} \; t_2 \neq t_3$$

The quantity t_o (generally not equal to zero) is the time at which the target vehicle is closest to the origin (the starting point for the intercept vehicle). The quantities t_1, t_2, t_3 are used in Chart I to indicate roots of the appropriate equations above, when they are known to exist. Figure 5(a) illustrates the various critical instants.

The above facts are used in Chart I to determine the logic necessary to locate the solution t^o and to determine whether a CATCH $[A(t^o) = E(t^o)]$ or a MISS $[\min[A(t^o)-E(t^o)]^2]$ is possible. Since the A equations change for $t > t_b$, t_b^- is used to indicate that the equations for $t < t_b$ should be used and vice versa for t_b^+. The times t_1, t_2, t_3 and t^o may all be found by using Newton's method for locating the zeros of a function. In the chart, f is used to indicate the function whose zero is sought. Since some of the functions have multiple zeros (and/or their derivatives have zeros), the symbol t_s is used to indicate a starting point for Newton's method. Due to monotone properties of the functions and derivatives involved, use of t_s as a starting point guarantees the convergence of Newton's method.

There are times when it might be desirable to let the intercept vehicle fire at some instant later than

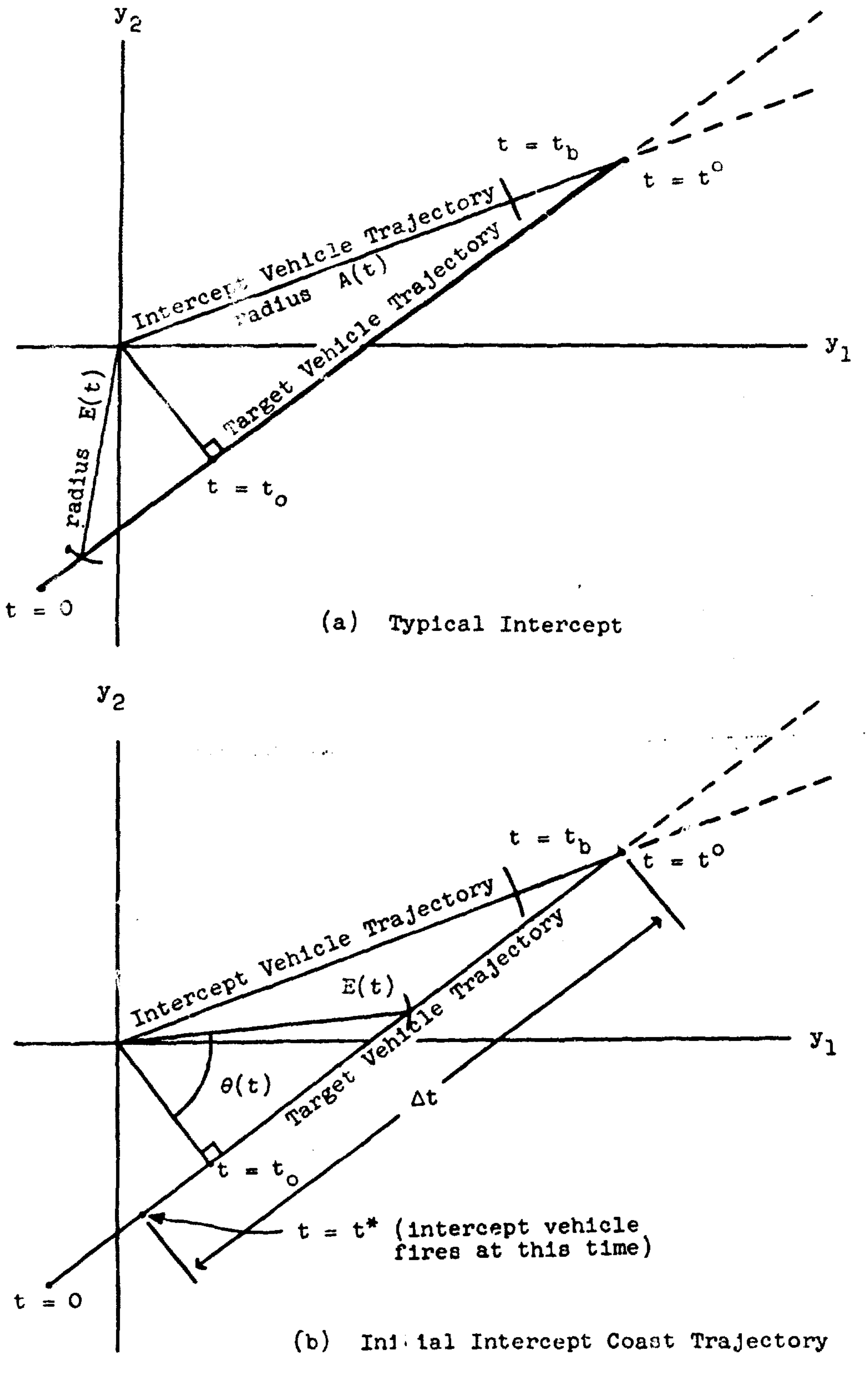

Figure 5. Geometrical Configuration of Intercept Trajectories.

zero, especially if the target is approaching ($t_o > 0$), see Figure 5(b). There is generally a maximum permissible period of initial coast, defined as t^* shown in Figure 5(b). The remaining material of this appendix deals with the analysis of this case. In order for a CATCH to be possible $A(t^o - t^*) = E(t^o)$. Differentiating with respect to t^o, one obtains

$$\dot{A}(t^o - t^*)\left[1 - \frac{dt^*}{dt^o}\right] = \dot{E}(t^o)$$

or

$$\frac{dt^*}{dt^o} = \dot{A}(t^o - t^*) - \dot{E}(t^o)$$

In order for t^* to be maximum $\frac{dt^*}{dt^o} = 0$; hence two necessary conditions for the determination of t^o and t^* are

$$A(t^o - t^*) = E(t^o)$$
$$\dot{A}(t^o - t^*) = \dot{E}(t^o)$$

The problem of determining t^o and t^* can be simplified if $E(t)$ and $\dot{E}(t)$ are expressed in terms of $E_o \triangleq E(t_o)$, minimum distance of the radius of target from the intercept origin, $V \triangleq \sqrt{x_3^2(0) + x_4^2(0)}$, the speed of the target vehicle, and $\theta(t)$, the angle between the position of the target vehicle at t_o and its position at any later time t, with $\theta(t)$ restricted to the first quadrant. Then

$$E(t) = E_o/\cos\,\theta(t)$$
$$\tan\,\theta(t) = V[t - t_o]/E_o$$
$$[\sec^2\theta(t)]\,\frac{d\theta(t)}{dt} = V/E_o$$
$$\dot{E}(t) = E_o\,\sin\,\theta(t)\,\sec^2\theta(t)\,\frac{d\theta(t)}{dt}$$
$$= V\,\sin\,\theta(t)$$

If $\Delta t \triangleq t^o - t^*$, the problem has become one of determining Δt and $\theta(t^o)$ such that

$$A(\Delta t) = E_o/\cos\,\theta(t^o)$$
$$\dot{A}(\Delta t) = V\,\sin\,\theta(t^o)$$

when $\theta(t^o)$ is eliminated, the following equation is

obtained

$$\left[\frac{\dot{A}(\Delta t)}{V}\right]^2 + \left[\frac{E_o}{A(\Delta t)}\right]^2 = 1$$

If

$$F(t) \triangleq [\dot{A}(t)A(t)]^2 + [VE_o]^2 - [VA(t)]^2$$

then

$$\dot{F}(t) = 2A(t)\dot{A}(t)[\ddot{A}(t)A(t) + [\dot{A}(t)]^2 - V^2]$$

and Δt is the root of the equation $F(\Delta t) = 0$. Since

$$\lim_{\Delta t \to 0}\left\{\left[\frac{\dot{A}(\Delta t)}{V}\right]^2 + \left[\frac{E_o}{A(\Delta t)}\right]^2\right\} = \infty > 1$$

$$\lim_{\Delta t \to \infty}\left\{\left[\frac{\dot{A}(\Delta t)}{V}\right]^2 + \left[\frac{E_o}{A(\Delta t)}\right]^2\right\} = \left[\frac{\dot{A}(t_b)}{V}\right]^2$$

it is easy to see that, if $\dot{A}(t_b) < V$, then $F(t)$ has at least one positive real zero. If $F(t_b) < 0$, then the zero must be less than t_b ($F(0) = [VE_o]^2 > 0$). If $F(t_b) > 0$, then with $F(t)$ rewritten as

$$F(t) = [VA(t)]^2\left\{\left[\frac{\dot{A}(t_b)}{V}\right]^2 - 1 + \left[\frac{E_o}{A(t)}\right]^2\right\}, \qquad t \geq t_b$$

the limit of $F(t)$ as $t \to \infty$ is clearly negative, and hence $F(t)$ has a zero for $t > t_b$. If $\dot{A}(t_b) > V$, the intercept vehicle can fire at any time and always catch the target because its burnout velocity is greaterr than the velocity of the target. Chart II indicates the logic steps necessary to locate the desired zero of $F(t)$.

Once Δt is determined, t^o is specified by $E(t^o) = A(\Delta t)$ with t^o chosen as the largest root of the equation. If $t^* = t^o - \Delta t < 0$, then interception is impossible and the intercept vehicle should fire at $t = 0$.

CHART I

Index	Possibility	Implication	Zero	of f =	t_s	MISS	CATCH	EXPLANATION
1.	$A(t_o)<E(t_o)$	$t^o>t_o$						
1.1	$t_o<t_b$							
1.1.1	$A(t_b)<E(t_b)$							
1.1.1.1	$\dot{A}(t_b)<\dot{E}(t_b)$							
1.1.1.1.1	$\dot{E}(0)\geq 0$	$t^o=0$				x		$\dot{A}\neq\dot{E}$ **but only possible solution**
1.1.1.1.2	$\dot{E}(0)<0$	$t_o<t_2<t_b$	t_2	$\dot{A}-\dot{E}$	t_o			$\ddot{A}-\ddot{E}$ monotonically increasing-f concave
1.1.1.1.2.1	$A(t_2)<E(t_2)$	$t^o=t_2$				x		
1.1.1.1.2.2	$A(t_2)>E(t_2)$	$t_o<t^o<t_2$	t^o	A-E	t_o		x	
1.1.1.2	$\dot{A}(t_b)>\dot{E}(t_b)$							
1.1.1.2.1	$\dot{E}(0)\geq 0$							
1.1.1.2.1.1	$\dot{A}(t_b)>\dot{E}(\infty)$	$t^o>t_b$	t^o	A-E	t_b^+		x	
1.1.1.2.1.2	$\dot{A}(t_b)<\dot{E}(\infty)$	$t_o<t_2<t_b<t_3$	t_3	$\dot{A}-\dot{E}$	t_b^+			t_2 must represent max $(A-E)^2$
1.1.1.2.1.2.1	$A(t_3)<E(t_3)$	$t^o=t_3$				x		
1.1.1.2.1.2.2	$A(t_3)>E(t_3)$	$t_b<t^o<t_3$	t^o	A-E	t_b^+		x	
1.1.1.2.2	$\dot{E}(0)<0$							Possibly three zeros of $(\dot{A}-\dot{E})$
1.1.1.2.2.1	$\ddot{A}(t_o)<\ddot{E}(t_o)$							
1.1.1.2.2.1.1	$\ddot{A}(t_b^-)<\ddot{E}(t_b)$							
1.1.1.2.2.1.1.1	$\dot{A}(t_b)>\dot{E}(\infty)$	$t^o>t_b$	t^o	A-E	t_b^+		x	
1.1.1.2.2.1.1.2	$\dot{A}(t_b)<\dot{E}(\infty)$	$t_2>t_b$	t_2	$\dot{A}-\dot{E}$	t_b^+			
1.1.1.2.2.1.1.2.1	$A(t_2)<E(t_2)$	$t^o=t_2$				x		
1.1.1.2.2.1.1.2.2	$A(t_2)>E(t_2)$	$t_b<t^o<t_2$	t^o	A-E	t_b^+		x	
1.1.1.2.2.1.2	$\ddot{A}(t_b^-)>\ddot{E}(t_b)$	$t_o<t_1<t_b$	t_1	$\ddot{A}-\ddot{E}$	t_o			

CHART I (Continued)

Index	Possibility	Implication	Zero	of f =	t_s	MISS	CATCH	EXPLANATION
1.1.1.2.2.1.2.1	$\dot{A}(t_1)<\dot{E}(t_1)$	$t_o<t_2<t_1$	t_2	$\dot{A}-\dot{E}$	t_o			
1.1.1.2.2.1.2.1.1	$A(t_2)<E(t_2)$							
1.1.1.2.2.1.2.1.1.1	$\dot{A}(t_b)>\dot{E}(\infty)$	$T^o>t_b$	t^o	A-E	t_b^+		x	
1.1.1.2.2.1.2.1.1.2	$\dot{A}(t_b)<\dot{E}(\infty)$	$t_3>t_b$	t_3	$\dot{A}-\dot{E}$	t_b^+			
1.1.1.2.2.1.2.1.1.2.1	$A(t_3)<E(t_3)$	$t^o=t_2$				x		or $t^o=t_3$ if $\lvert A(t_3)-E(t_3)\rvert<\lvert A(t_2)-E(t_2)\rvert$
1.1.1.2.2.1.2.1.1.2.2	$A(t_3)>E(t_3)$	$t_b<t^o<t_3$	t^o	A-E	t_b^+		x	
1.1.1.2.2.1.2.1.2	$A(t_2)>E(t_2)$	$t_o<t^o<t_2$	t^o	A-E	t_o		x	
1.1.1.2.2.1.2.2	$\dot{A}(t_1)>\dot{E}(t_1)$	$t_2>t_b$						same as 1.1.1.2.2.1.1
1.1.1.2.2.2	$\ddot{A}(t_o)>\ddot{E}(t_o)$	$t_2>t_b$						same as 1.1.1.2.2.1.1
1.1.2	$A(t_b)>E(t_b)$	$t_o<t^o<t_b$						must catch
1.1.2.1	$\dot{A}(t_b)<\dot{E}(t_b)$		t^o	A-E	t_o		x	
1.1.2.2	$\dot{A}(t_b)>\dot{E}(t_b)$							
1.1.2.2.1	$\dot{E}(0)\geq 0$	$t_o<t^o<t_b$	t^o	A-E	t_b		x	
1.1.2.2.2	$\dot{E}(0)<0$							Possibly two roots of $(\dot{A}-\dot{E})$ in (t_o,t_b)
1.1.2.2.2.1	$\ddot{A}(t_o)<\ddot{E}(t_o)$							
1.1.2.2.2.1.1	$\ddot{A}(t_b)<\ddot{E}(t_b)$	$t_o<t^o<t_b$	t^o	A-E	t_o		x	
1.1.2.2.2.1.2	$\ddot{A}(t_b)>\ddot{E}(t_b)$	$t_o<t_1<t_b$	t_1	$\ddot{A}-\ddot{E}$	t_o			
1.1.2.2.2.1.2.1	$\dot{A}(t_1)<\dot{E}(t_1)$	$t_o<t_2<t_1$	t_2	$\dot{A}-\dot{E}$	t_o			
1.1.2.2.2.1.2.1.1	$A(t_2)<E(t_2)$	$t_1<t^o<t_b$	t^o	A-E	t_b		x	
1.1.2.2.2.1.2.1.2	$A(t_2)>E(t_2)$	$t_o<t^o<t_2$	t^o	A-E	t_o		x	
1.1.2.2.2.1.2.2	$\dot{A}(t_1)>\dot{E}(t_1)$							
1.1.2.2.2.1.2.2.1	$A(t_1)>E(t_1)$	$t_1<t^o<t_b$	t^o	A-E	t_b		x	

CHART I (Continued)

Index	Possibility	Implication	Zero of f =	ts	MISS	CATCH	EXPLANATION
1.1.2.2.2.1.2.2.2	$A(t_1)>E(t_1)$	$t_o<t^o<t_1$	t^o A-E	t_o		x	
1.1.2.2.2.2	$\ddot{A}(t_o)>\ddot{E}(t_o)$	$t_o<t^o<t_b$	t^o A-E	t_o		x	
1.2	$t_o>t_b$						
1.2.1	$\dot{A}(t_b)>\dot{E}(\infty)$	$t^o>t_o$	t^o A-E	t_o		x	
1.2.2	$\dot{A}(t_b)<\dot{E}(\infty)$	$t_2>t_o$	t_2 $\dot{A}$-$\dot{E}$	t_o			
1.2.2.1	$A(t_2)<E(t_2)$	$t^o= t_2$			x		
1.2.2.2	$A(t_2)>E(t_2)$	$t_o<t^o<t_2$	t^o A-E	t_o		x	
2.	$A(t_o)>E(t_o)$	$t^o<t_o$					always catch
2.1	$t_o<t_b$	$t^o<t_o$	t^o A-E	t_o		x	
2.2	$t_o>t_b$						
2.2.1	$A(t_b)<E(t_b)$	$t_b<t^o<t_o$	t^o A-E	t_o		x	
2.2.2	$A(t_b)>E(t_b)$	$t^o<t_b$	t^o A-E	$t_{\bar{b}}$		x	

CHART II

Index	Possibility	Implication	Zero	of f =	ts	MISS	CATCH	EXPLANATION
1.	$F(t_b)>0$							
1.1	$\dot{F}(t_{\bar{b}})>0$	$0<t_2<t_b$	t_2	$\dot{F}$	$t_b/2$			
1.1.1	$F(t_2)<0$	$0<t_1<t_2$	t_1	F	$t_2/2$			
		$t_2<t_1<t_b$	t_1	F	$t_{\bar{b}}$			choose $\Delta t=t_1$ yielding
		$t_1>t_b$	t_1	F	t_b^+			largest $t^* = t^o - \Delta t$
1.1.2	$F(t_2)>0$	$t_1>t_b$	t_1	F	t_b^+			$\Delta t = t_1$
1.2	$F(t_{\bar{b}})<0$	$t_1>t_b$	t_1	F	t_b^+			$\Delta t = t_1$
2.	$F(t_b)<0$							
2.1	$\dot{F}(t_{\bar{b}})>0$	$0<t_1<t_b$	t_1	F	$t_b/2$			$\Delta t = t_1$
2.2	$\dot{F}(t_{\bar{b}})<0$	$0<t_1<t_b$	t_1	F	$t_{\bar{b}}$			$\Delta t = t_1$

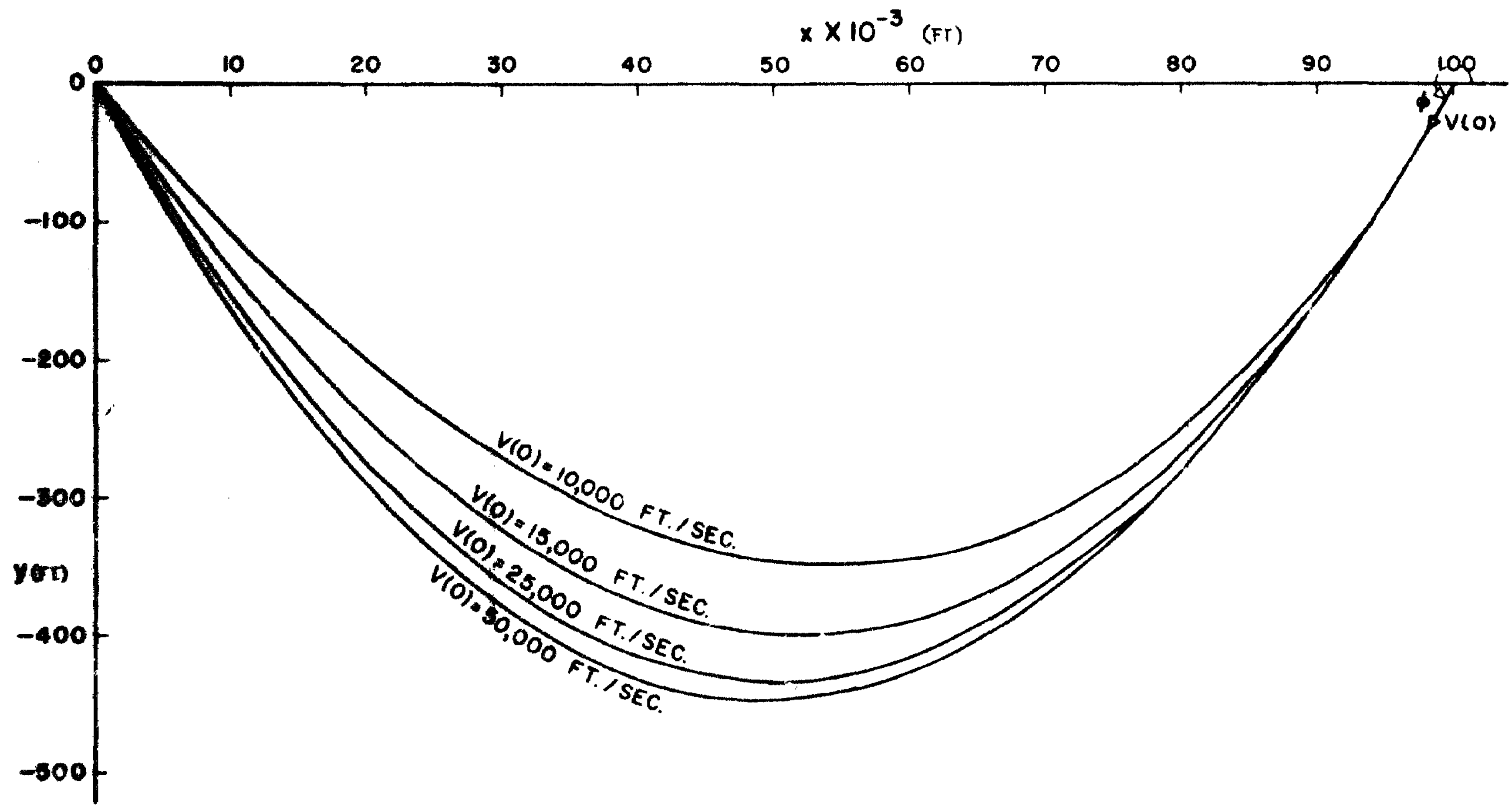

Figure 6. High Velocity Intercept Paths, Rocket #1, $\phi = 181^{\circ}$, $\rho_o = 1745$ ft., $R(0) = 100,000$ ft.

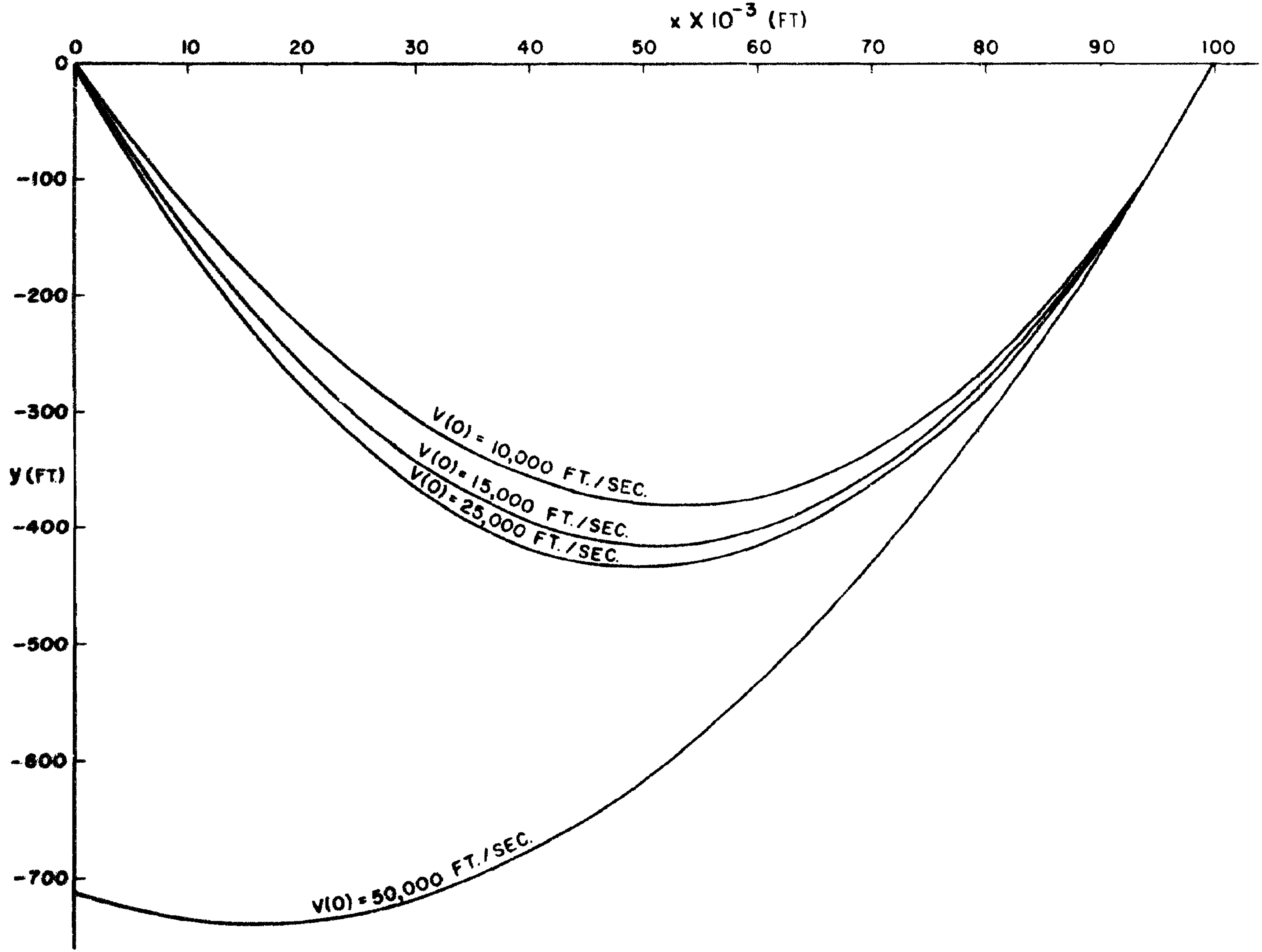

Figure 7. High Velocity Intercept Paths, Rocket #2, $\phi = 181^{\circ}$, $\rho_o = 1745$ ft., R(0) = 100,000 ft.

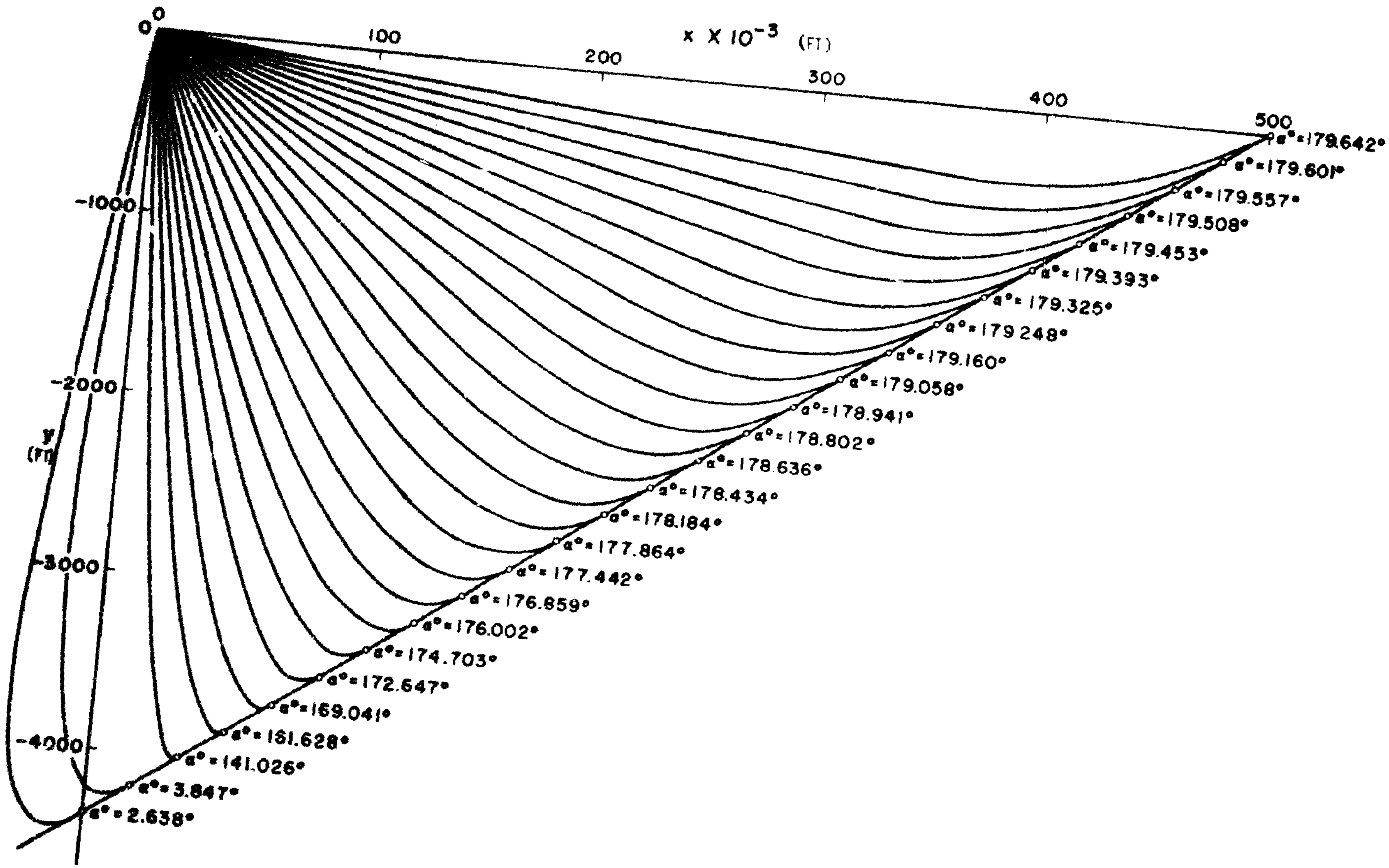

Figure 8. Family of Interception Paths, Rocket #1, $\phi = 180.5^{\circ}$, $\rho_o = 4363$ ft., $R(0) = 500,000$ ft., $V(0) = 10,000$ ft./sec.

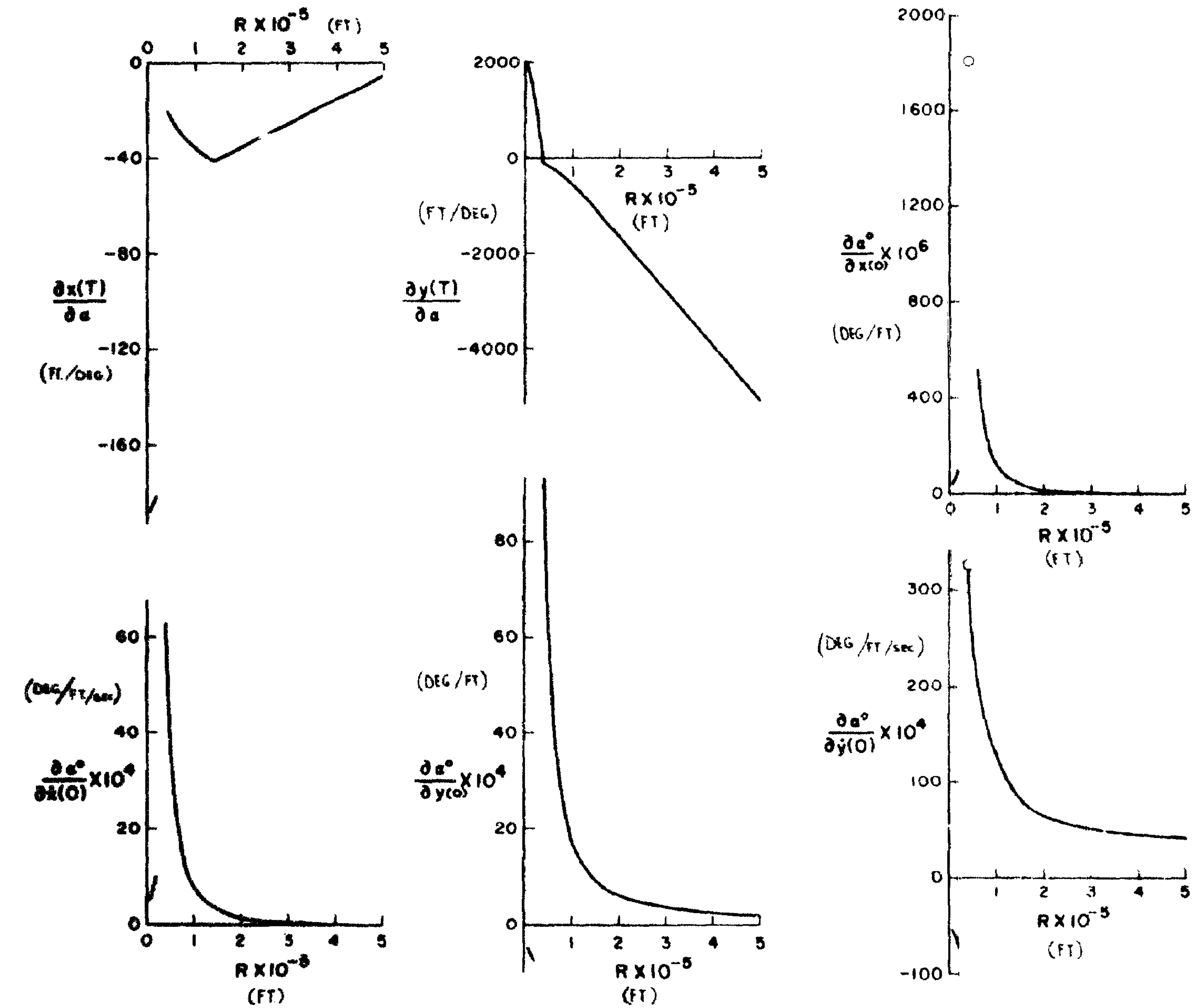

Figure 9. Terminal Error Sensitivity to Initial Measurement and Steering Errors, Rocket #1, V(0) = 10,000 ft./sec.

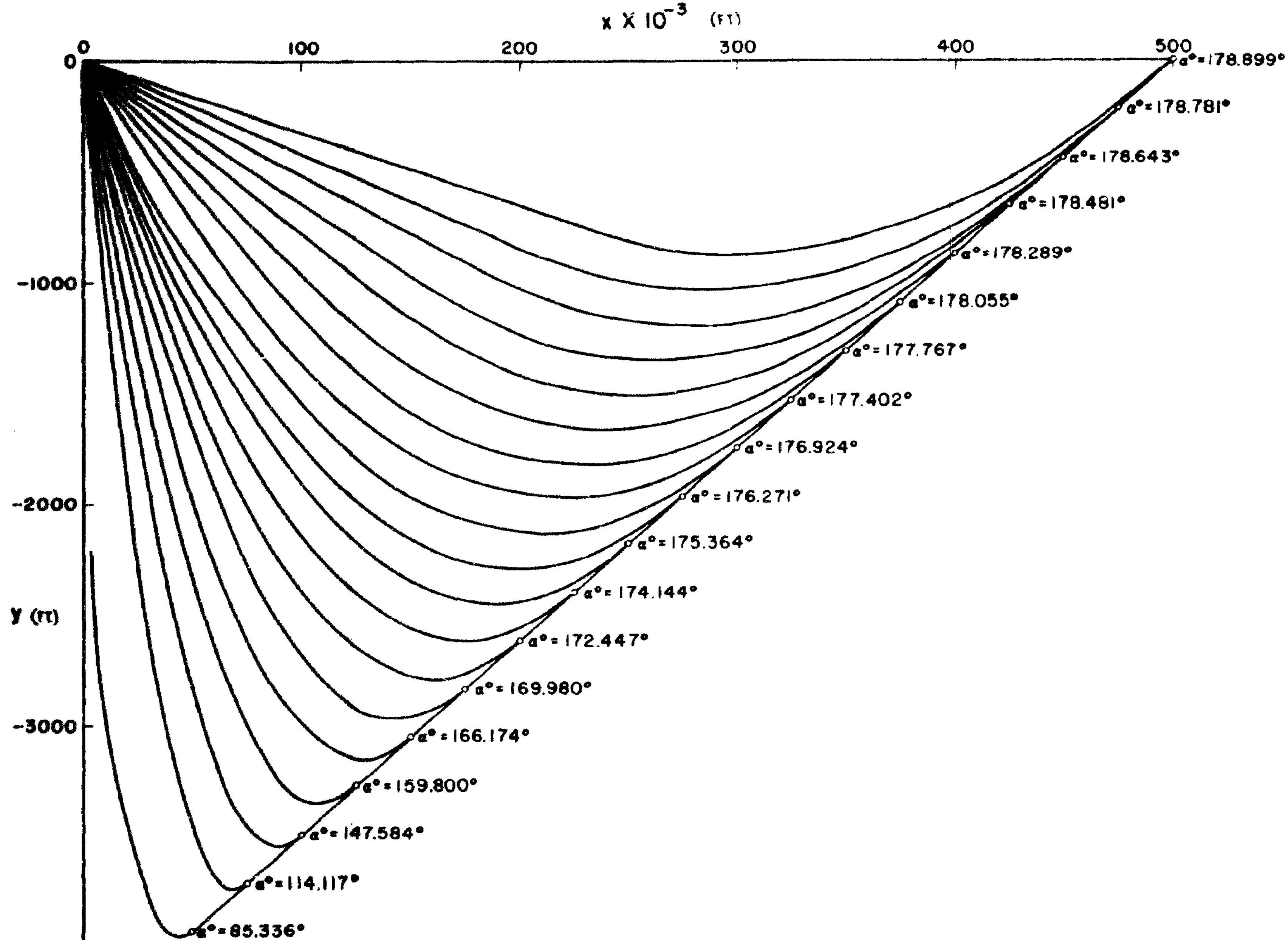

Figure 10. Family of Interception Paths, Rocket #1, $\phi = 180.5^{\circ}$, $\rho_o = 4363$ ft., $R(0) = 500{,}000$ ft., $V(0) = 25{,}000$ ft./sec.

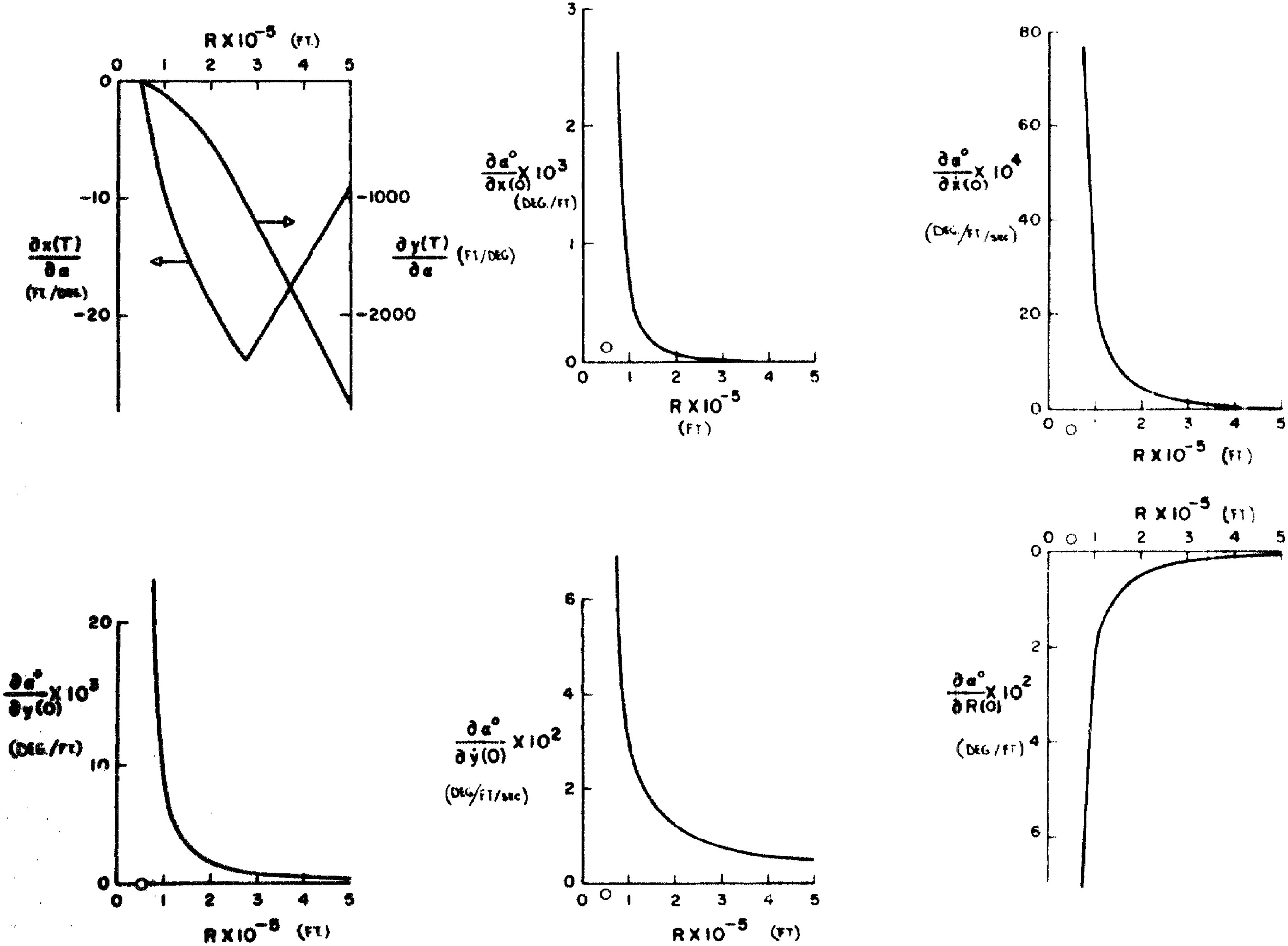

Figure 11. Terminal Error Sensitivity to Initial Measurement and Steering Errors, Rocket #1, V(0) = 25,000 ft./sec.

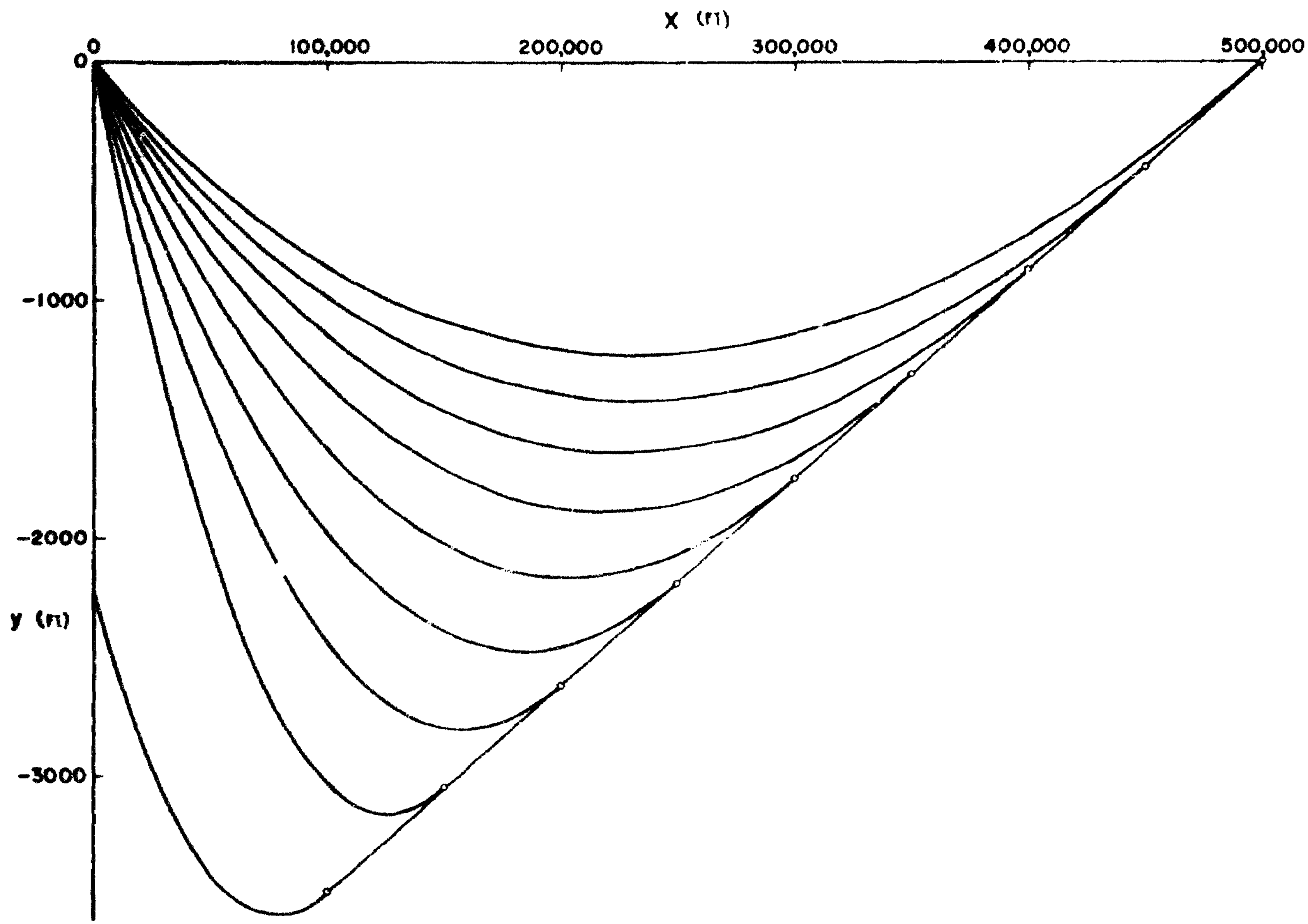

Figure 12. Family of Interception Paths, Rocket #1, $\phi = 180.5^{\circ}$, $\rho_o = 4363$ ft., $R(0) = 500,000$ ft., $V(0) = 50,000$ ft./sec.

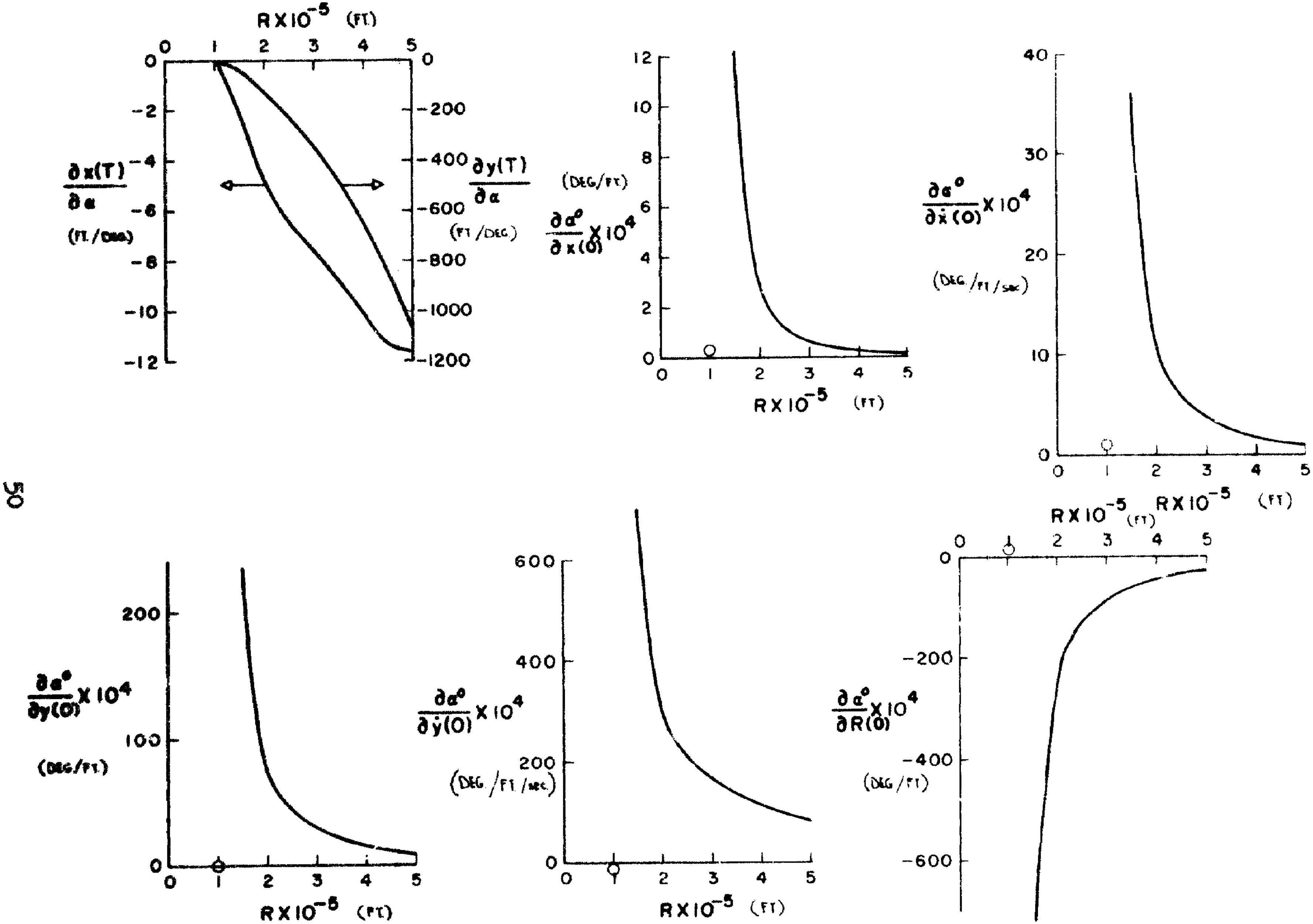

Figure 13. Terminal Error Sensitivity to Initial Measurement and Steering Errors, Rocket #1, V(0) = 50,000 ft./sec.

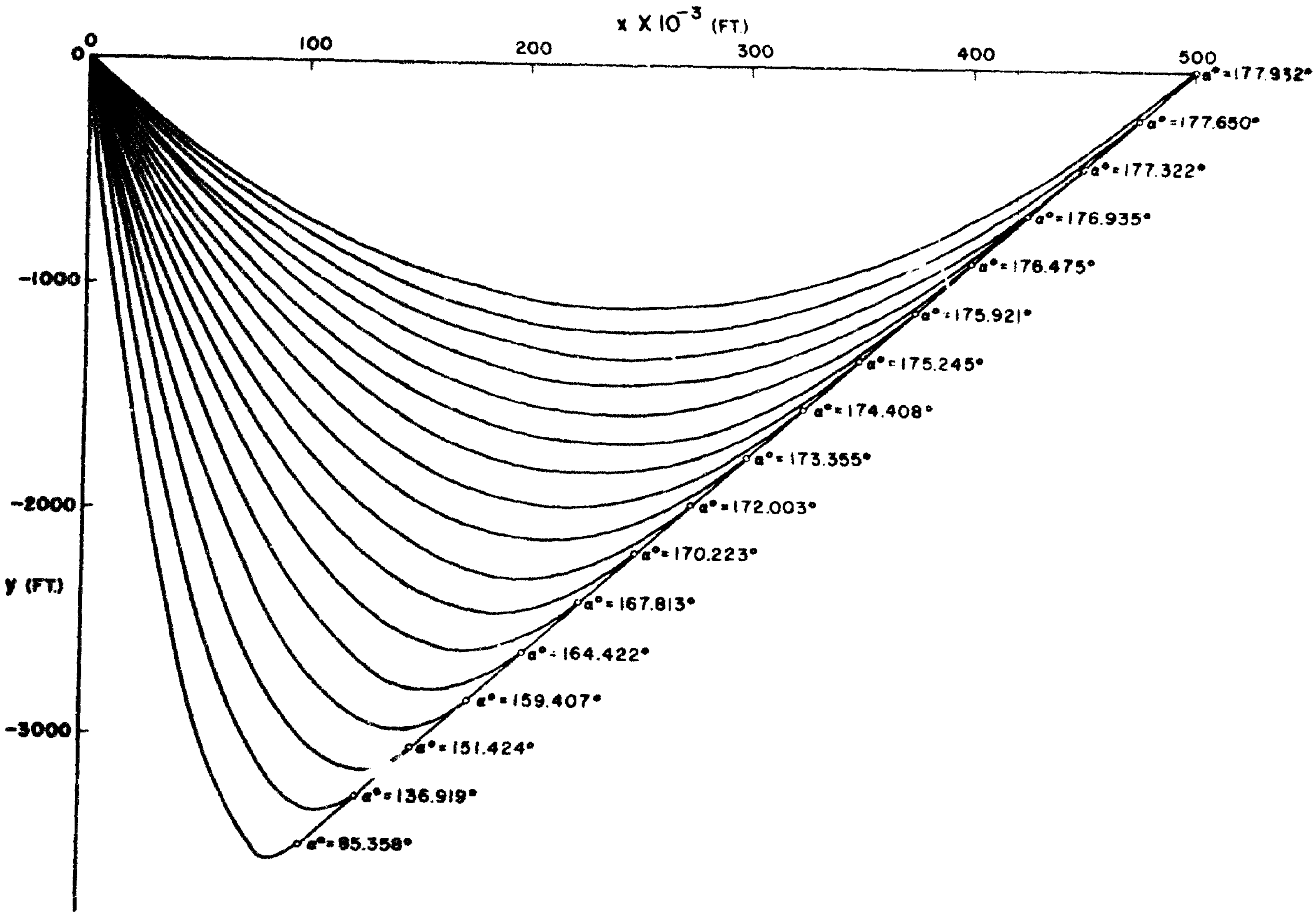

Figure 14. Family of Interception Paths, Rocket #2, ϕ = 180.5°, ρ_o = 4363 ft., R(0) = 500,000 ft., V(0) = 25,000 ft./sec.

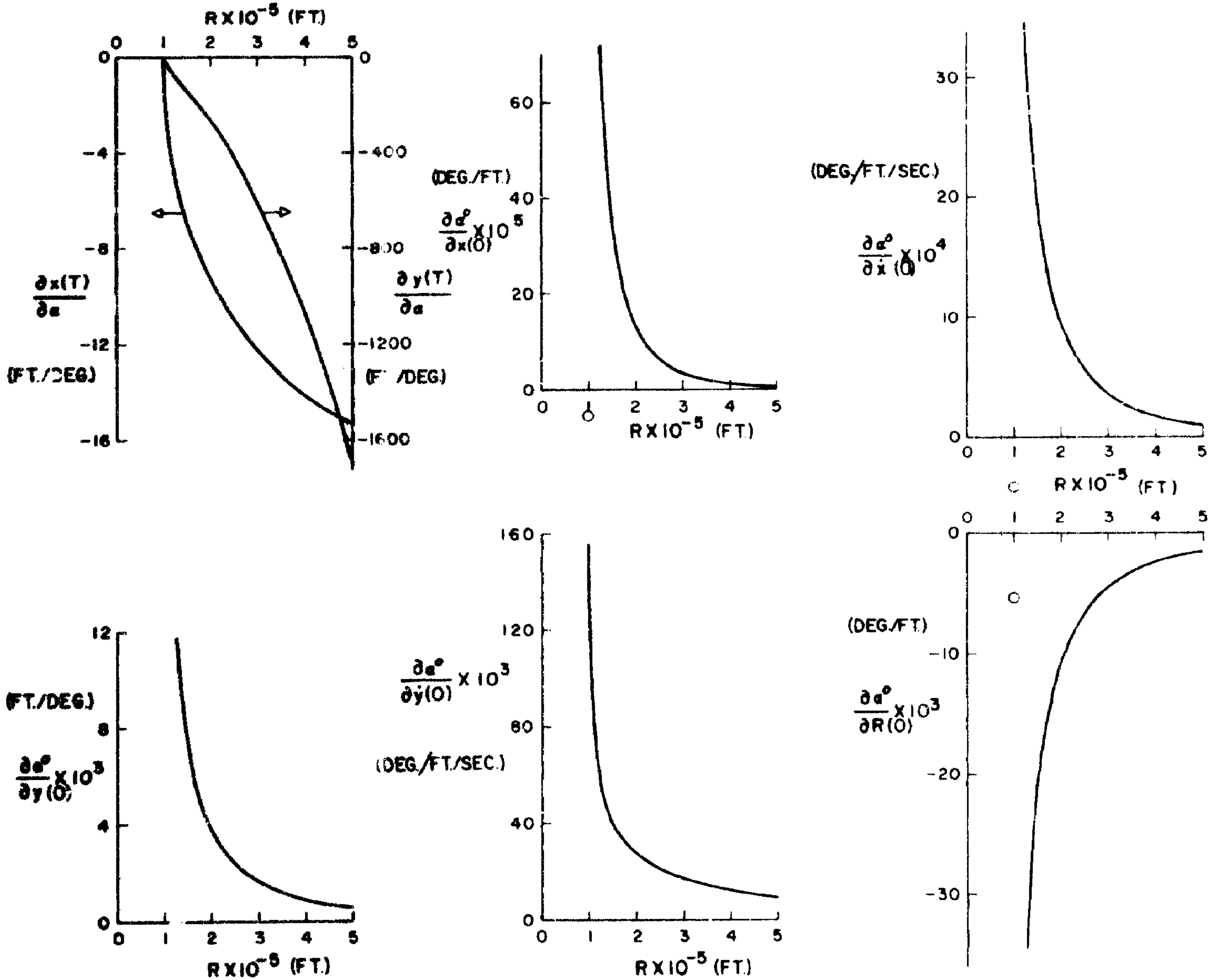

Figure 15. Terminal Error Sensitivity to Initial Measurement and Steering Errors, Rocket #2, V(0) = 25,000 ft./sec.

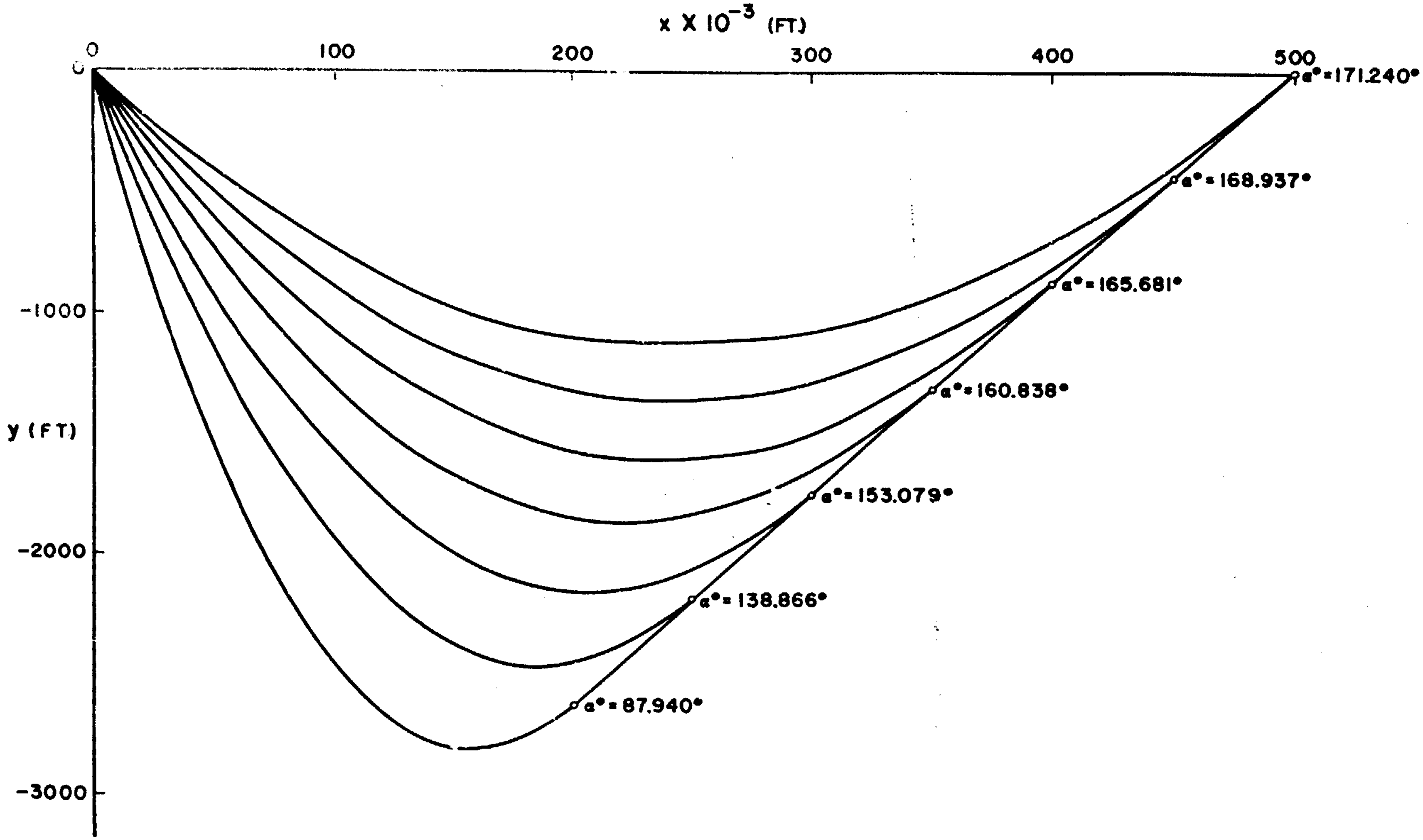

Figure 16. Family of Interception Paths, Rocket #2, $\phi = 180.5^{\circ}$, $\rho_o = 4363$ ft., $R(0) = 500{,}000$ ft., $V(0) = 50{,}000$ ft./sec.

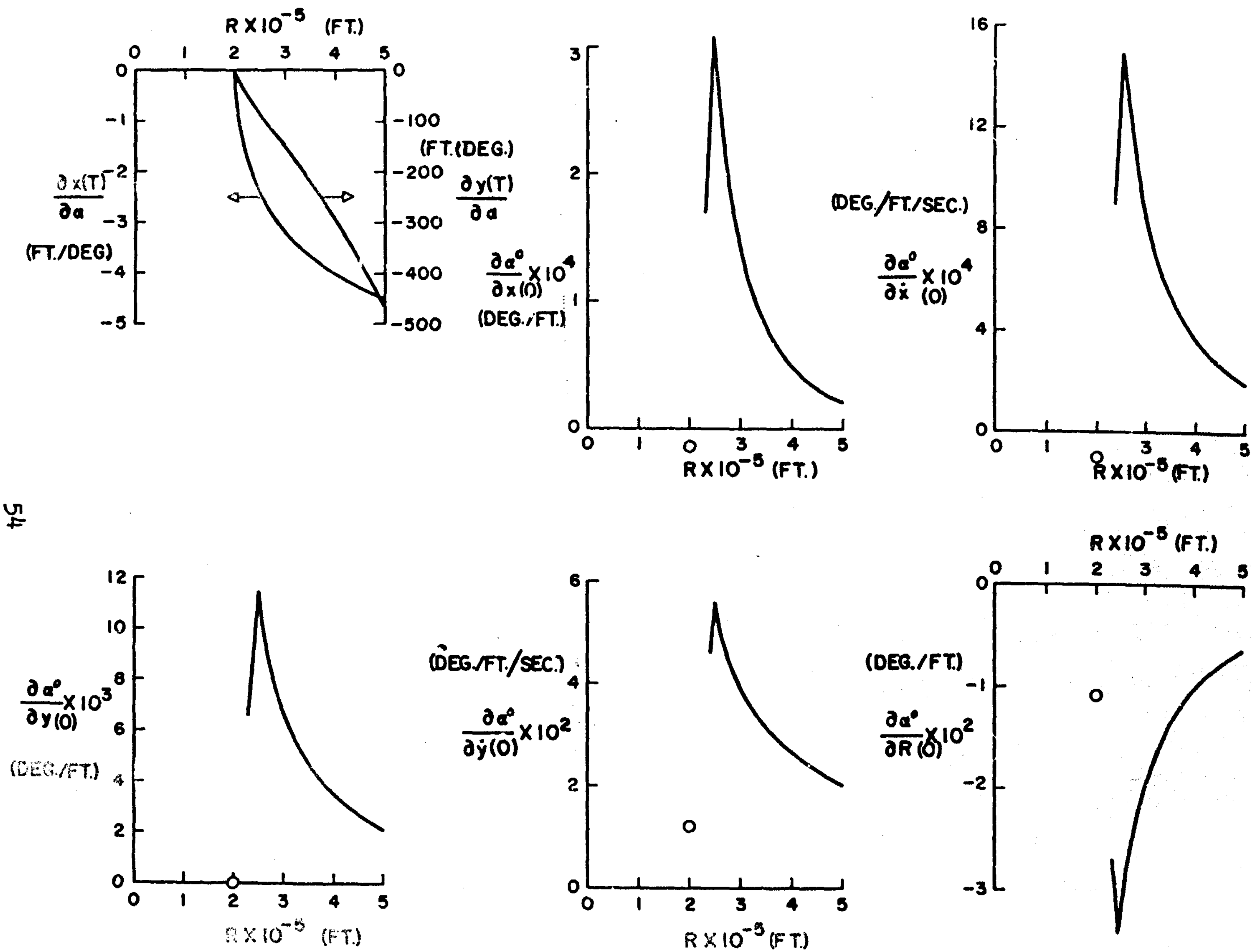

Figure 17. Terminal Error Sensitivity to Initial Measurement and Steering Errors, Rocket #2, V(0) = 50,000 ft./sec.

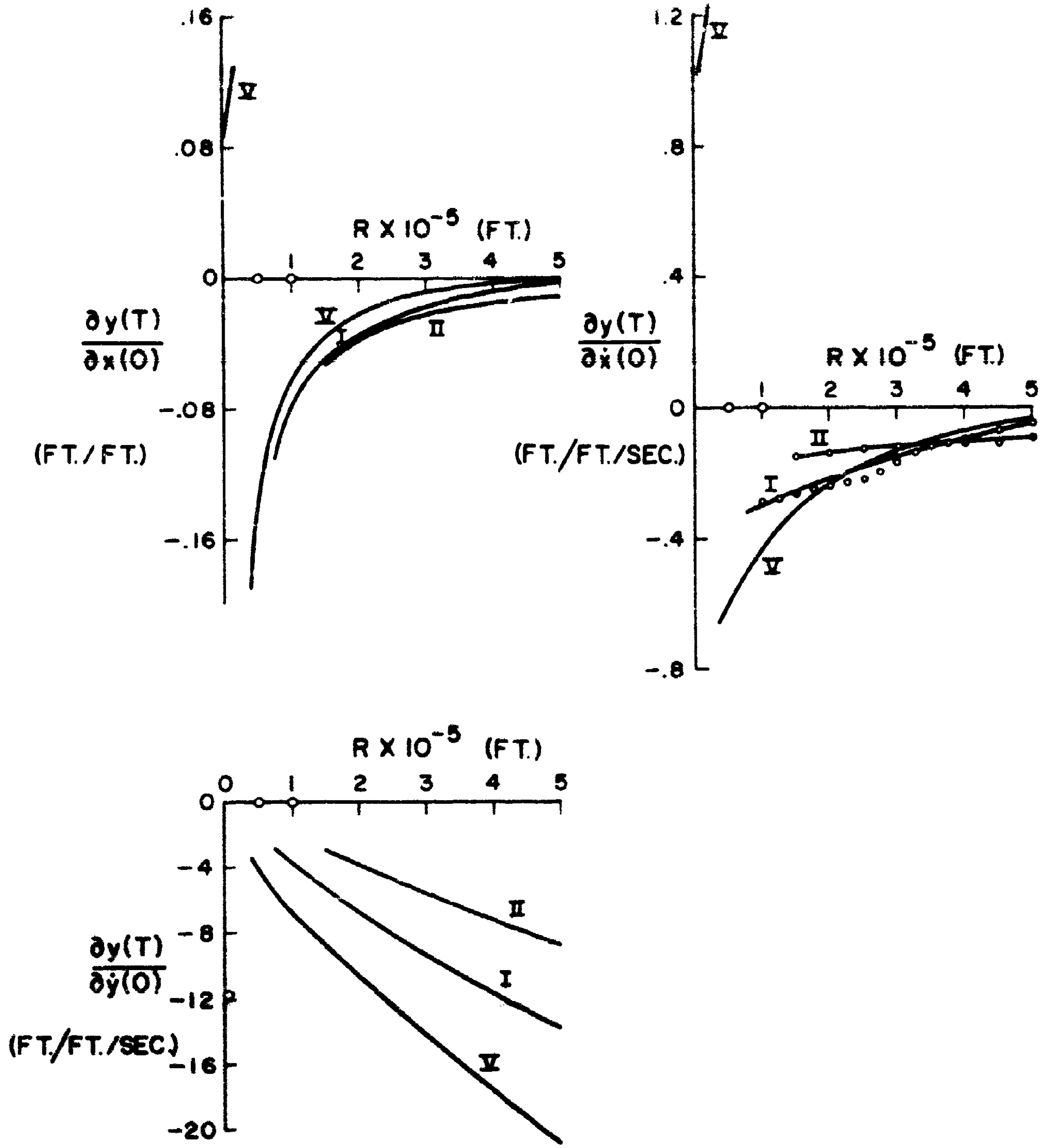

Figure 18. Summary, Rocket #1, ϕ = 180.5°, ρ_0 = 4363 ft., R(0) = 500,000 ft., V(0) = I - 25,000, II - 50,000, V - 10,000 ft./sec.

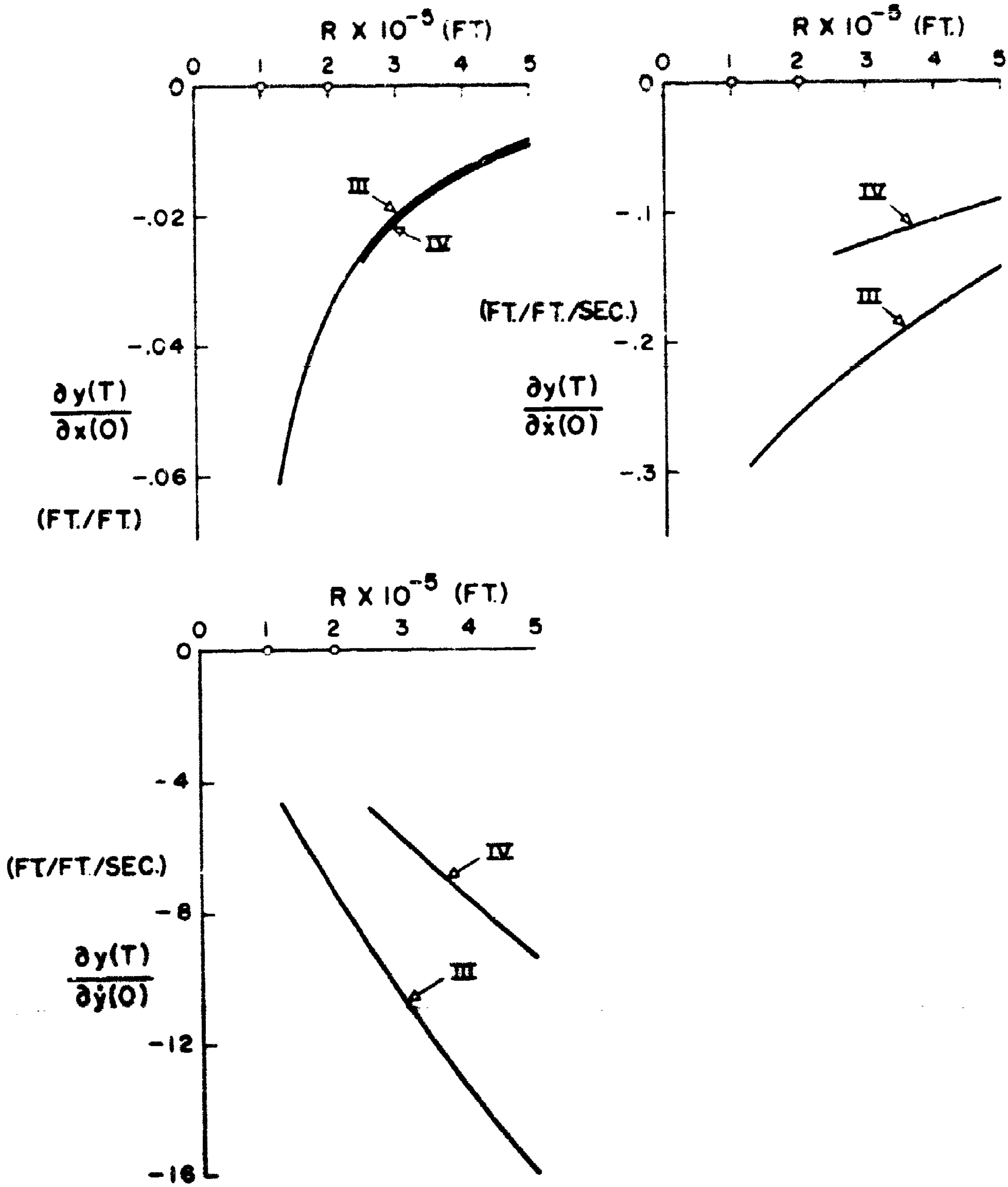

Figure 19. Summary, Rocket #2, $\phi = 180.5^{\circ}$, $\rho_o = 4363$ ft., $R(0) = 500{,}000$ ft., $V(0) =$ III - 25,000, IV - 50,000 ft./sec.

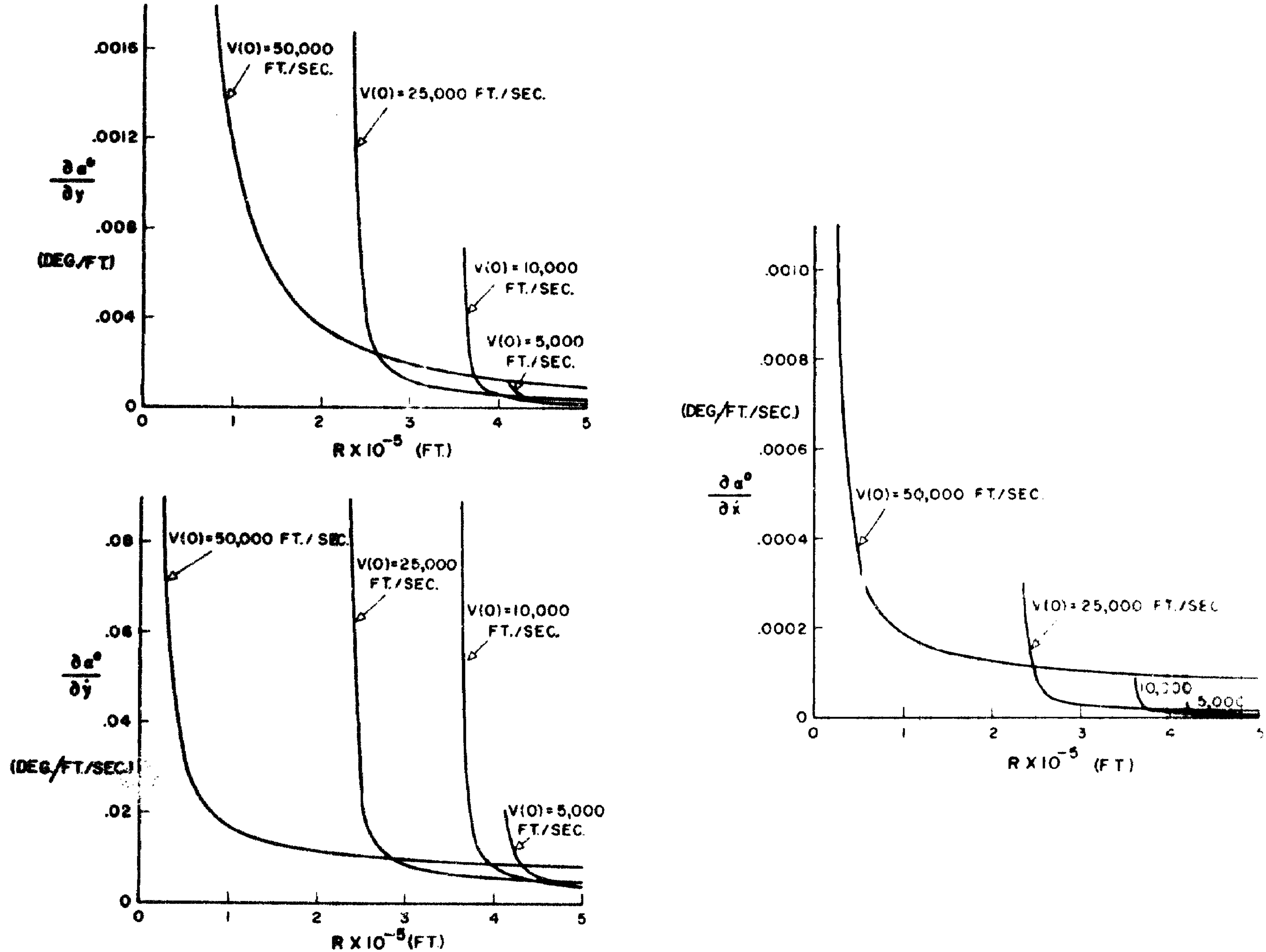

Figure 20. Summary Error Coefficient Along Optimal Path, Rocket #1

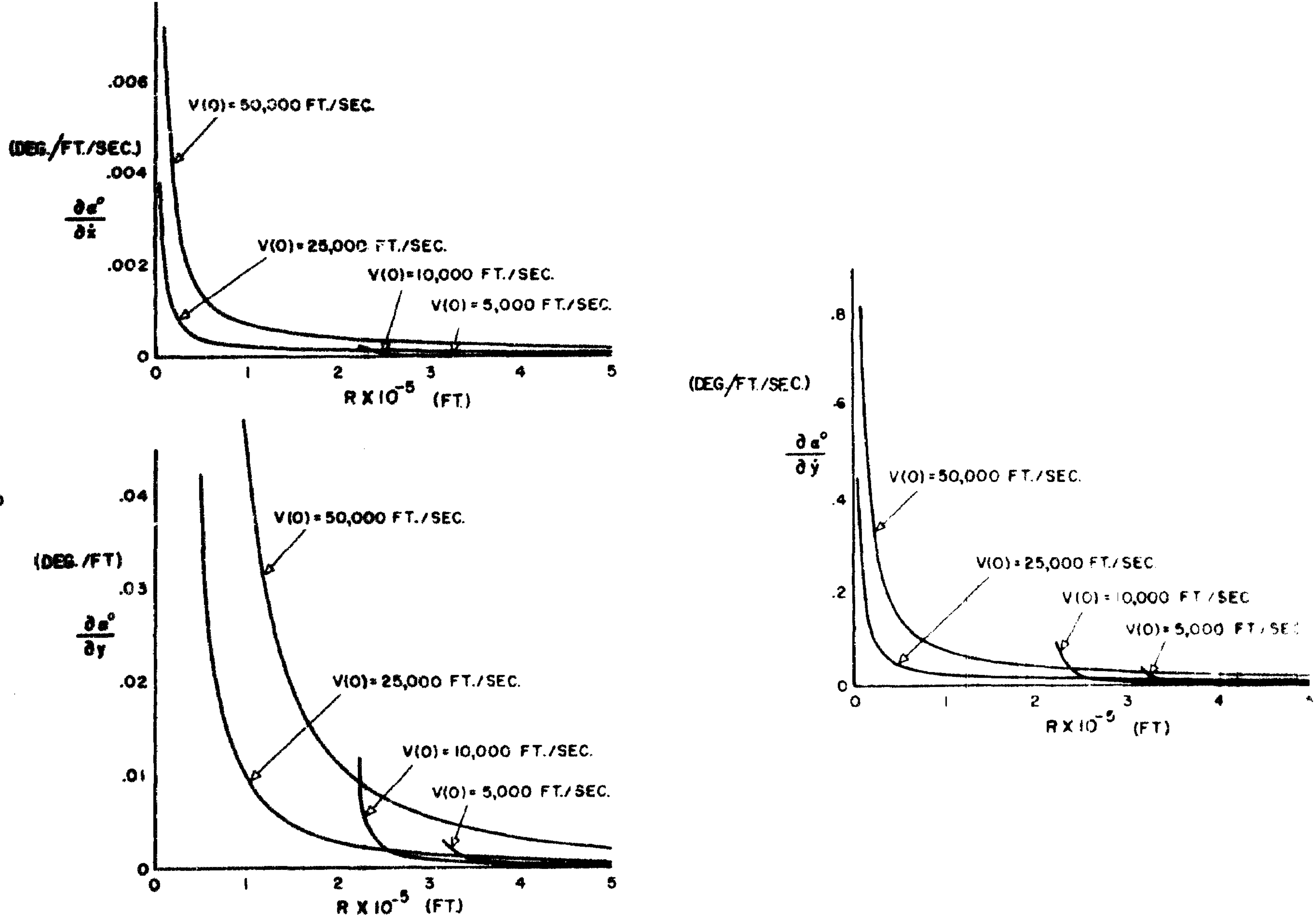

Figure 21. Summary Error Coefficient Along Optimal Path, Rocket #2

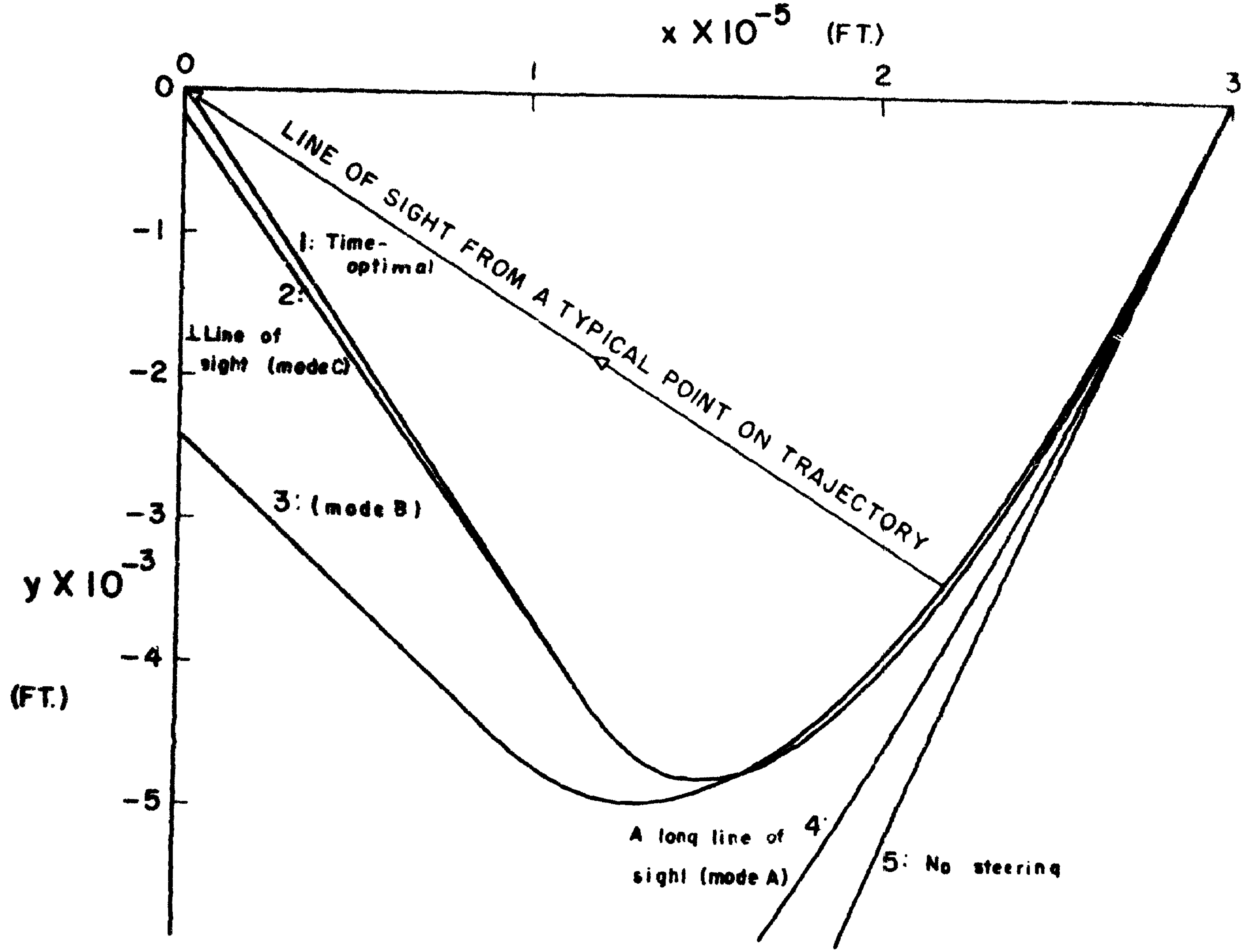

Figure 22. Interception with Optimal and Proportional Guidance, [illegible], R(0) = 300,000 ft., V(0) = 25,000 ft./sec.

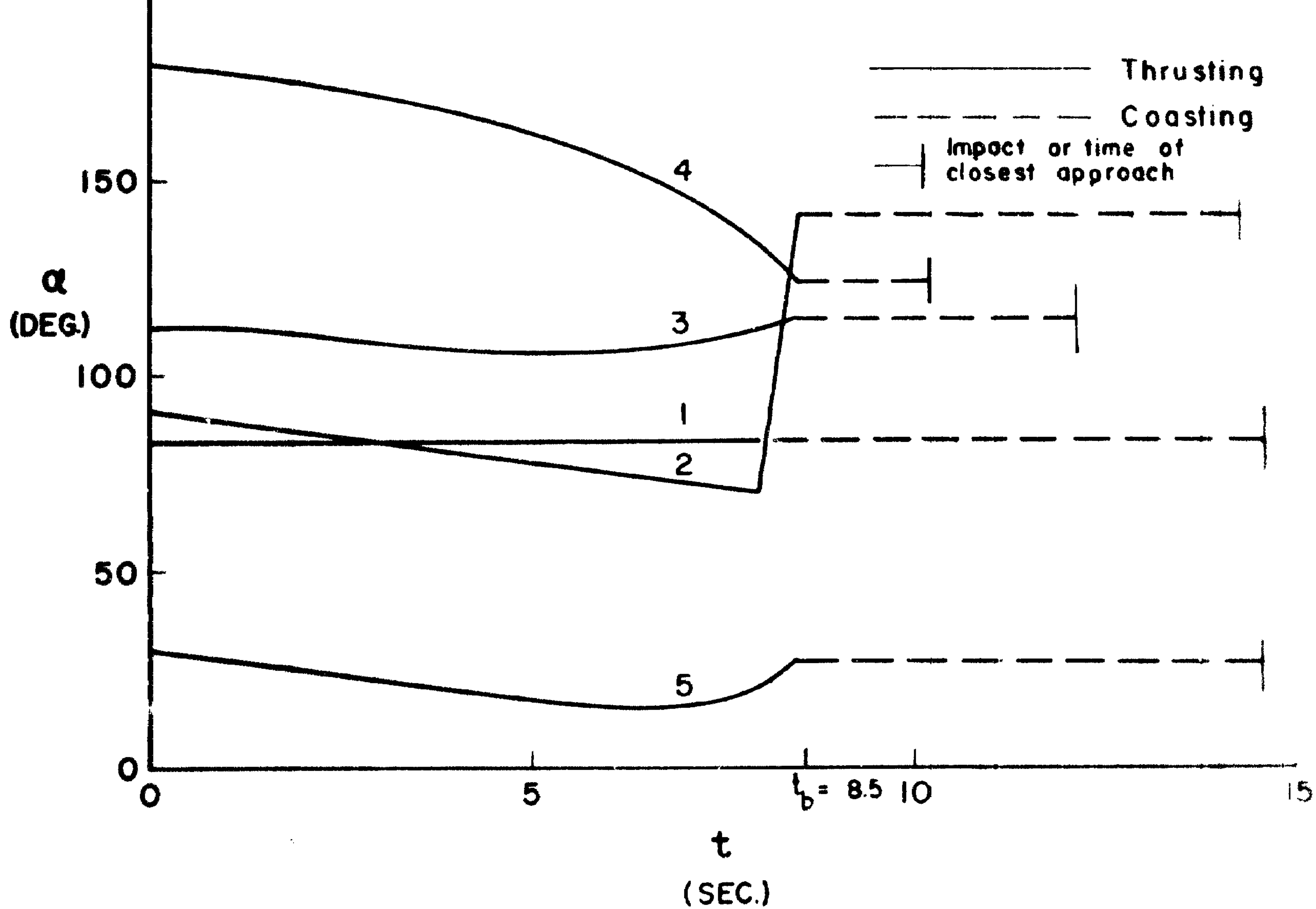

Figure 23. Steering Angles vs. Time for Proportional [illegible]

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13. ABSTRACT

A planar time-optimal formulation is used to study the terminal phase of interception above the atmosphere.in a uniform gravitational field. The dependence on initial conditions of the optimal steering angle is examined for initial relative velocities ranging from 5,000 ft/sec to 50,000 ft/sec and initial distances up to 500,000 ft. Results are presented in graphical form for two typical rockets showing (1) the range dependence of terminal error sensitivities to errors in measurements of initial conditions taken along the initial rectilinear coasting path and (2) the variation in these terminal error coefficients along optimal interception paths. Some interceptions with proportional control systems were made and compared with the optimal paths.